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## Epimorphisms between Groups given by means of Generators and Defining Relations

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EPIMORPHISMS BETWEEN GROUPS GIVEN  
BY MEANS OF GENERATORS AND  
DEFINING RELATIONS

by

Alfred Dominic Vella B Sc.

A thesis presented to the

Open University

for the degree of

Doctor of Philosophy in Group Theory

Milton Keynes

July, 1977.

FOR CAROL

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## STATEMENT

Of §§1.1, 1.2 and 1.4 only the treatment of pieces using the 'piece function' is original. In the remaining sections results of other workers are given due credit.

A handwritten signature in black ink, appearing to read 'A Vella'.

A. VELLA

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## ABSTRACT

Chapter One is expository.

In Chapter Two we consider the following questions.

Let  $G = \langle X ; R \rangle$ .

Is  $G$  Hopfian?

Is  $\text{Aut}(G)$  finitely generated?

Is there an algorithm to decide for any two words  $W_1, W_2$  in  $X$  whether or not  $\{W_1, W_2\}$  generates  $G$ ?

Let  $J$  be a set of group presentations. Is there an algorithm to decide whether or not two elements of  $J$  define isomorphic groups?

The study of these questions concerns the study of endomorphisms between groups given by means of generators and defining relations. They have been considered by Pride in cases when  $G$  is a one-relator group with torsion. Using methods similar to those of Pride we obtain positive results for certain other groups particularly small cancellation groups. A problem related to Hopficity (stability) is also studied in Chapter Two.

In Chapter Three we study two questions. The first is. Let  $G = \langle X ; R \rangle$  and suppose that all elements of  $R$  can be expressed in the free group  $F$  on  $X$  as a set of freely reduced words  $T$  in a set of words  $W$  in  $X$ . Does  $\text{sgp}(W)$  have presentation  $\langle h ; T(h) \rangle$  under the mapping  $h \mapsto W$ ?

We look at this question in detail and give many positive as well as negative results.



The other question in Chapter Three concerns malnormality. It is known that a subset of the generators of a one-relator group with torsion generates a malnormal subgroup. Is this result true for small cancellation groups? We show that the answer to this question is no in general but yes if all the relators are proper powers greater than two.

## NOTE TO THE READER

The discussion in §1.1 gives the problems to be studied in this thesis and the broad strategy used to tackle these problems. In §1.2 we give an indication of the extent of our knowledge of these problems, excluding the work done in the thesis. New results obtained in this thesis are surveyed in §1.3. These three sections together give an overall view of the whole thesis. In §1.4 we give the necessary background results of small cancellation theory which are used in later chapters. Chapters Two and Three are independent of each other, but depend upon Chapter One.

## NOTATION AND DEFINITIONS

Sets are usually denoted by Gothic letters, e.g.  $R$ . We will adopt the usual notation of set theory. If  $A$  and  $B$  are sets,  $A \setminus B$  will denote the set of elements of  $A$  which are not elements of  $B$ ;  $A \Delta B$  will denote the set of elements in either  $A$  or  $B$ , but not in both. The cardinal of a set  $A$  will be denoted by  $|A|$ . It is sometimes useful to treat a set as an ordered tuple, and vice versa. This we do whenever expedient, and no confusion is likely. The elements of a set  $A$  will be denoted by  $A_i$ ,  $a_i$  or  $a$ . By  $A \subset B$  we mean that  $A$  is a proper subset of  $B$ . We sometimes use  $B \supset A$  for  $A \subset B$ . For ease of notation we write  $a$  for  $\{a\}$ . If  $V$  and  $W$  are sets with  $|V| = |W|$ , then  $V = W$  denotes the set of equations  $V_i = W_i$  for some fixed ordering of  $V$  and  $W$ . The context should make it clear which ordering is meant. If  $X$  is a set, we say that  $W$  is a *word in  $X$*  if it is a word in the elements of  $X$ .

Undefined concepts and notation will be as in Magnus, Karrass and Solitar [1966], which will be referred to as MKS from now on.

Let  $G$  be a group. The *commutator*  $g^{-1} h^{-1} gh$  of two elements  $g$  and  $h$  of  $G$  will be denoted by  $[g, h]$ . We denote the commutator subgroup of  $G$  by  $G'$ . Two elements  $g, h$  of  $G$  are *conjugate* if there is an  $f$  in  $G$  with  $f^{-1} g f = h$ , and we write  $g \sim h$ . If  $W$  is a subset of  $G$ , then  $\text{sgp}_G W$  will denote the subgroup of  $G$  generated by the elements of  $W$ . The  $G$  in  $\text{sgp}_G$  is omitted if it is clear what  $G$  should be. If  $W$  is a subset of  $G$  and  $a$  is an element of  $G$ ,  $\text{sgp}\{W, a\}$  is the subgroup of  $G$  generated by  $W \cup \{a\}$ : other similar abuses of notation will be used.

When a group  $G$  is given in terms of generators  $X$  and defining relators, words in  $X$  will often be identified with the elements of  $G$  which they define. The context should make it clear when this identification is being made. In particular, if  $W$  is a set of words in  $X$  then  $\text{sgp } W$  will denote the subgroup of  $G$  generated by a set of elements of  $G$  defined by the words in  $W$ , and we write  $\langle W \rangle^G$  for the normal closure of  $W$  in  $G$ . The identity of any group, and the subgroup it generates will be denoted by  $1$ .

The group  $G$  will be called an  $r$ -generator group if and only if it has a subset  $W$  of cardinality  $r$ , such that  $\text{sgp } W = G$  (for some ordering of  $W$ ).  $W$  is then called a *generating  $r$ -tuple* of  $G$ .

Let  $G = \langle X ; R \rangle$ , and let  $V, W$  be words in the generators  $X$  of  $G$ .

$V \equiv W$  will mean that  $V$  and  $W$  are the same word;

if  $V$  and  $W$  define the same element of  $G$ , then it will be said that  $V$  is equal to  $W$  in  $G$ , written  $V = W$  in  $G$ ,  
 $V = W$  or just  $V = W$  if  $G$  is understood;

$V$  and  $W$  will be said to be *freely equal* if and only if  $V$  can be transformed into  $W$  by a finite sequence of insertion and deletions of pairs of the form  $xx^{-1}$  and  $x^{-1}x$ , where  $x$  is an element of  $X$ ;

$V$  will be said to be *freely reduced* if and only if it has no subwords of the form  $x^{-1}x$  or  $xx^{-1}$  for  $x \in X$ , and will be said to be *cyclically reduced* if and only if all cyclic permutations of it are freely reduced.

Suppose  $G = \langle X ; R \rangle$  and that  $T$  is a set. We write  $T(X)$  to indicate that  $T$  is a set of words in  $X$ . By  $T = 1$  we mean that each element of  $T$  defines the identity in  $G$ .

The number of symbols in the word  $W$  will be called the *length* of  $W$ , and be denoted by  $\ell(W)$ . A word  $V$  will be said to be *more than half* of  $W$  if and only if  $V$  is a subword of  $W$  and  $\ell(V) > \frac{1}{2}\ell(W)$ . Concepts such as *less than half*, *exactly half* etc. are defined in an analogous manner. We say that  $W$  *contains*  $V$  if  $V$  is a subword of  $W$ .

If  $W$  is a word in  $X$ , then  $x \in X$  will be said to *occur with exponent*  $\alpha$  in  $W$  if and only if  $W \equiv Px^\alpha Q$ , where neither the last symbol of  $P$  nor the first symbol of  $Q$  is  $x$  or  $x^{-1}$ . If  $x$  occurs with exponent different from zero in  $W$  then it will be said that  $W$  *involves*  $x$ , or equivalently that  $x$  *occurs in*  $W$ . If  $x$  occurs with exponents  $\alpha_1, \dots, \alpha_n$  in  $W$  then the *exponent sum of  $x$  in  $W$*  is  $\sum_{i=1}^n \alpha_i$ , and is written  $\sigma_x(W)$ .

The set of generators occurring in a word  $W$  is written  $\delta(W)$ .

If  $X$  is a set,  $F(X)$  denotes the free group on  $X$ . We will denote by  $F_n$  a free group of rank  $n$ ; the context will make it clear what the generators of  $F_n$  are.

A mapping  $\phi$  from a group  $G$  into a group  $H$  will usually be given by its action on a generating tuple of  $G$ , and by  $\phi(g_1, \dots, g_n)$  we will mean  $(\phi g_1, \dots, \phi g_n)$ . The kernel of  $\phi$  will be written  $\ker \phi$ . Let  $X, g$  be sets with  $|X| = |g|$ . By  $X \mapsto g$  we mean the mapping taking  $x_i$  to  $g_i$  for some fixed ordering of  $X$  and  $g$ .

If  $G$  and  $H$  are two groups then  $G * H$  denotes the free product of  $G$  and  $H$ . If  $g$  is a subset of  $G$  and  $h$  is a subset of  $H$  with  $|g| = |h|$  then  $G \underset{g=h}{*} H$  or  $G \underset{g \ h^{-1}}{*} H$  will denote the generalised free product of  $G$  and  $H$  amalgamating  $spg \ g$  with  $spg \ h$ , under the mapping  $g \mapsto h$ .

We will also use the notation  $H \underset{h_i = g_i}{*} G_i$  to mean the generalised free product of groups  $H$  and  $G_i$  with  $\cup_i h_i$  amalgamated with  $\cup_i g_i$  under the mapping  $h_i \mapsto g_i$ , where  $h_i \in H$  and  $g_i \in G_i$ .

Throughout the thesis  $\varepsilon, \varepsilon'$  and variations of these denote integers of modulus 1. We will also need the *lexographical ordering* of  $r$ -tuples of integers. We say that  $(n_1, n_2, \dots, n_r)$  is *less than*  $(m_1, m_2, \dots, m_r)$ , if whenever  $i$  is the smallest  $j$  for which  $n_j \neq m_j$ , then  $n_i < m_i$ . Minimality of  $r$ -tuples of integers will always be with respect to this ordering, subject to any other stated restrictions.

If  $\alpha, n$  and  $m$  are integers the fact that  $\alpha$  divides  $m$  will be written  $\alpha | m$  and if  $n - m$  is a multiple of  $\alpha$  we write  $m = n \bmod \alpha$ .

The following notations are introduced in the text. The numbers in brackets refer to the pages where the notations are introduced.

$C'(\lambda)$	small cancellation condition(19).
$C(p)$	small cancellation condition(20).
$\hat{C}(p)$	small cancellation condition(60).
$p$	the 'piece' function(21).
$R^*$	symmetrized set(21).
$R_*$	(28).
$\phi^*$	induced automorphism(5).

## CHAPTER ONE

### §1.1 The Problems.

Two problems of interest to group theorists are the following.

Let  $G$  be a group.

A. Under what conditions is  $G$  Hopfian? [Recall that a group is said to be *Hopfian* if all its endomorphisms are automorphisms, otherwise it is *non-Hopfian*.]

B. Under what conditions does  $G$  have a finitely generated, or even finitely presented, automorphism group?

In this thesis we will be concerned with the above questions in the case when  $G$  is given in terms of generators and defining relations.

A third problem in which we will be interested is the isomorphism problem.

C. Given a set  $J$  of group presentations, is there an algorithm to decide whether or not any two elements of  $J$  define isomorphic groups?

The isomorphism problem for  $J$  will be denoted by  $\text{ISOP}(J)$ , and we say that  $\text{ISOP}(J)$  is *solvable* if such an algorithm exists.

One feature in common to all these problems is that they concern homomorphisms from one group onto another (where the groups may coincide). Let  $\rho$  be a homomorphism from a group  $G_1$  onto a group  $G_2$ .

Then  $\rho$  must take every generating set of  $G_1$  onto a generating set of  $G_2$ . Thus, work on the above questions leads us to a study of generators.

Let  $F$  denote the free group freely generated by  $X$  and let  $G$  be a  $|X|$  - generator group. Suppose that  $g$  and  $g'$  are generating  $|X|$  - tuples of  $G$ . Then  $g$  and  $g'$  are said to be *Nielsen equivalent* if there is an automorphism  $X \mapsto Y(X)$  of  $F$ , such that  $g' = Y(g)$ . Also,  $g$  and  $g'$  are said to *lie in the same T-system* if there is an automorphism  $\theta$  of  $G$  such that  $g'$  is Nielsen equivalent to  $\theta(g)$ .

T-systems are related in a natural way to presentations. Let  $g$  be a generating  $|X|$  - tuple of  $G$  and let  $R \subseteq F$ . Then  $\langle X ; R \rangle$  is a presentation of  $G$  associated with  $g$  if and only if the kernel of the mapping  $F \rightarrow G$  given by  $X \mapsto g$  is  $\langle R \rangle^F$ .

LEMMA. If  $\langle X ; R \rangle$  is a presentation of  $G$  associated with the  $|X|$  - tuple  $g$ , then  $g'$  lies in the same T-system as  $g$  if and only if there is an automorphism  $\phi$  of  $F$  such that  $\langle X ; \phi(R) \rangle$  is a presentation of  $G$  associated with  $g'$ .

Proof. If  $g$  and  $g'$  lie in the same T-system, then there are automorphisms  $\theta, \phi^{-1}$  of  $G$  and  $F$  respectively, with  $\phi^{-1} : X \mapsto Y(X)$  and  $g' = Y(\theta g)$ . Let  $\alpha : F \rightarrow G$  be given by  $X \mapsto g$ . Then the kernel of  $\alpha$  is  $\langle R \rangle^F$ . Let  $\beta : F \rightarrow G$  be given by  $X \mapsto g'$ . Then it is easily seen that  $\beta = \theta \alpha \phi^{-1}$ , and so the kernel of  $\beta$  is  $\langle \phi(R) \rangle^F$ .

Conversely suppose  $\langle X ; R \rangle$  and  $\langle X ; \phi(R) \rangle$  are presentations of  $G$  associated with  $g$  and  $g'$  respectively, where  $\phi$  is an automorphism of  $F$ . We will show that  $g$  and  $g'$  lie in the same T-system. Let



$\phi : X \mapsto Z(X)$ ,  $\phi^{-1} : X \mapsto Y(X)$ , and consider the diagram:

$$\begin{array}{ccc} & F & \xrightarrow{\alpha} G \\ \phi^{-1} \downarrow & \searrow \gamma & \\ F & \xrightarrow{\beta} & G \end{array}$$

$$\alpha : X \mapsto g', \text{ Ker } \alpha = \langle \phi(R) \rangle^F$$

$$\beta : X \mapsto g, \text{ Ker } \beta = \langle R \rangle^F$$

Then  $\gamma : X \mapsto Y(g)$ . But  $\text{ker } \gamma = \text{Ker } \beta\phi^{-1} = \langle \phi(R) \rangle^F = \text{ker } \alpha$ , so there is an automorphism  $\theta$  of  $G$  such that  $\theta\gamma = \alpha$ . Thus  $g' = Y(\theta(g))$ , and so  $g$  and  $g'$  lie in the same T-system, as required.

||

It is not difficult to show (see Neumann and Neumann [1951]) that Nielsen equivalence is an equivalence relation, and so the class of generating  $|X|$  - tuples splits into disjoint classes called *Nielsen equivalence classes of generating  $|X|$  - tuples* (or *Nielsen equivalence classes* if  $X$  or  $|X|$  is understood). Lying in the same T-system is also an equivalence relation, and each T-system is the union of Nielsen equivalence classes.

The usefulness of T-systems and Nielsen equivalence classes in connection with Problems A - C stems from the following results.

PROPOSITION A. Suppose that all generating  $|X|$  - tuples of  $G$  lie in the same T-system as  $g$ , and that  $\langle X ; R \rangle$  is a presentation of  $G$  associated with  $g$ . Suppose also that  $\langle R \rangle^F$  has the property that whenever  $\langle R \rangle^F \subseteq \langle \phi(R) \rangle^F$  for an automorphism  $\phi$  of  $F$  then  $\langle \phi(R) \rangle^F = \langle R \rangle^F$ . Then  $G$  is Hopfian.

Proof. Let  $\gamma$  be an endomorphism of  $G$ . We will show that  $\gamma$  is an automorphism. Since  $\gamma$  is onto,  $\gamma g$  generates  $G$ , and so lies in the same T-system as  $g$ . By the Lemma, there is an automorphism  $\phi$  of  $F$  such that  $\langle X ; \phi(R) \rangle$  is a presentation of  $G$  associated with  $\gamma g$ . Let  $\alpha : F \rightarrow G$  be defined by  $X \mapsto g$ , and  $\beta : F \rightarrow G$  be defined by  $X \mapsto \gamma g$ .

Then the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & G \\ & \searrow \beta & \downarrow \gamma \\ & & G \end{array}$$

Clearly  $\ker \alpha \subseteq \ker \gamma\alpha$ . Moreover  $\ker \alpha = \langle R \rangle^F$  and  $\ker \gamma\alpha = \ker \beta = \langle \phi(R) \rangle^F$ . Thus  $\ker \alpha = \ker \gamma\alpha$  (by the conditions imposed on  $\langle R \rangle^F$ ), and so  $\gamma$  is an automorphism, as required.

||

We call a set  $R$  satisfying the conditions in the proposition *stable*, otherwise it is *unstable*. This extends the definition given in Pride [1976b], where 'stable' is defined for normal subgroups only.

Note that strictly speaking we must give the free group in which we are working before stability is meaningful, for it is conceivable that a set  $R$  may be stable in one free group but unstable in another. I would conjecture, however, that this cannot happen, but as yet have no proof of this, except when  $R$  consists of a single element of  $F$ . When we say that  $R$  is stable, the context should make it clear which free group is intended.

PROPOSITION B. Let  $\langle X ; R \rangle$  be a presentation of  $G$  associated with the generating  $|X|$  - tuple  $g$ , and suppose the  $T$ -system containing  $g$  is the union of finitely many Nielsen equivalence classes. Let  $\Phi = \{\phi : \phi \in \text{Aut}(F) : \langle \phi(R) \rangle^F = \langle R \rangle^F\}$ . If  $\Phi$  is finitely generated then  $\text{Aut}(G)$  is finitely generated.

Proof. Define a mapping  $*$  of  $\Phi$  into  $\text{Aut}(G)$  as follows. If  $\phi : X \mapsto Y(X)$  then  $\phi^*g = Y(g)$ . The mapping  $*$  is an antihomomorphism (see MKSp130), and so the image of  $\Phi$  under  $*$  is finitely generated.

Now let  $N_1, N_2, \dots, N_p$  be the distinct Nielsen equivalence classes whose union is the  $T$ -system containing  $g$ . Assume  $g \in N_1$ , and let  $g' \in N_i$ . Then there is a  $\phi \in \text{Aut } F$  defined by  $\phi: X \mapsto Y(X)$  and  $\theta_i \in \text{Aut}(G)$  such that  $g' = Y(\theta_i(g))$ . Since  $Y(\theta_i(g)) = \theta_i(Y(g))$  it follows that  $g'$  is the image under  $\theta_i$  of an element of  $N_1$ . Conversely any element of  $N_1$  maps onto an element of  $N_i$  under  $\theta_i$ .

Suppose  $\theta \in \text{Aut}(G)$ . Then  $\theta g \in N_i$  for some  $i$ , and so  $\theta_i^{-1}\theta \in N_1$ . Consequently  $\theta_i^{-1}\theta = \phi^*$  for some  $\phi \in \Phi$ . Hence the image of  $\Phi$  under  $*$  has finite index in  $\text{Aut}(G)$ , so  $\text{Aut}(G)$  must be finitely generated if  $\Phi^*$  is.

||

The next proposition concerns the isomorphism problem. To make the formulation simpler a few preliminary remarks will be made.

Let  $J$  be a recursive set of finite presentations of groups, and let  $x_1, x_2, \dots$  be a fixed but arbitrary countably infinite alphabet. Then if  $\langle Y ; S \rangle \in J$  where  $|Y| = n$ , one can replace the generating symbols in  $Y$  by  $x_1, x_2, \dots, x_n$  and make the corresponding changes to the relations  $S$ .

The above procedure gives a set of finite presentations on the fixed alphabet  $x_1, x_2, \dots$ . The procedure is clearly effective, so the isomorphism problem for the new set is solvable if and only if it is solvable for the original set  $J$ .

PROPOSITION C. Let  $J$  be a recursive set of finite presentations on a fixed alphabet  $x_1, x_2, \dots$  and suppose every element of  $J$  defines a group with one  $T$ -system. If there is an algorithm to decide for any pair  $\langle x_1, x_2, \dots, x_m; R \rangle, \langle x_1, x_2, \dots, x_n; S \rangle$  in  $J$ , with  $m \leq n$ , whether or not there is an automorphism of  $F_n$  such that  $\langle R \cup \{x_{m+1}, \dots, x_n\} \rangle^{F_n} = \langle \phi(S) \rangle^{F_n}$ , then  $\text{ISOP}(J)$  is solvable.

Proof. Suppose  $G = \langle x_1, \dots, x_m; R \rangle$  and  $H = \langle x_1, x_2, \dots, x_n; S \rangle$ , where  $\langle x_1, \dots, x_m; R \rangle$  and  $\langle x_1, \dots, x_n; S \rangle$  belong to  $J$ , and  $m \leq n$ . Let  $X = (x_1, \dots, x_n)$ ,  $\bar{R} = R \cup \{x_{m+1}, \dots, x_n\}$  and  $\bar{G} = \langle X; \bar{R} \rangle$ . Clearly  $G \cong \bar{G}$ . Now if  $H \cong \bar{G}$  then, since  $H$  has one  $T$ -system (of generating  $n$ -tuples), it follows from the Lemma that  $\langle \bar{R} \rangle^{F_n} = \langle \phi(S) \rangle^{F_n}$  for some  $\phi \in \text{Aut}(F_n)$ . Conversely, if  $\langle \bar{R} \rangle^{F_n} = \langle \phi(S) \rangle^{F_n}$  for some  $\phi \in \text{Aut}(F_n)$  then clearly  $H \cong \bar{G}$ . The result now follows immediately.

||

The study of generating sets leads us onto another question.

D. Let  $G = \langle X; R \rangle$  and suppose that all elements of  $R$  can be expressed in  $F$  as freely reduced words  $T$  in a set of words  $W$  in  $X$ . Does  $\text{sgp } W$  have presentation  $\langle h; T(h) \rangle$  under the map  $h \mapsto W$ ?

If it does we will say that  $\text{sgp } W$  has the obvious presentation.

A group  $G$  has a group  $H$  as factor if  $G \cong (H * K) / N$ , for some  $K$  and for some normal subgroup  $N$  of  $H * K$ , with  $H$  embedded into  $G$  by the natural map  $\nu : H * K \rightarrow G$ .

The relationship between this notion and Problem D is best illustrated by an example. Suppose  $H = \langle h ; T(h) \rangle$ ,  $K = F$  and  $N = \langle \{h_i, W_i^{-1} : i \in I\} \rangle^{H * K}$ . Then  $G = \langle X ; T(W) \rangle$ , and if  $G$  has  $H$  as a factor,  $\text{sgp}_G W$  has the obvious presentation. These considerations lead to the following proposition, the proof of which is almost immediate.

**PROPOSITION D.** *Let  $G = \langle X ; R \rangle$ , and suppose that all elements of  $R$  can be expressed in  $F$  as freely reduced words  $T$  in a set  $W$  of freely reduced words in  $X$ . Then  $\text{sgp}_G W$  has the obvious presentation if and only if  $G$  has  $\langle h ; T(h) \rangle$  as a factor, with  $h \mapsto W(X)$  under the natural map.*

||

We will also be looking briefly at the two-generator problem 2GP, which for a group  $G$  given by a presentation  $\langle X ; R \rangle$  is the algorithmic problem of deciding for any pair of words  $W_1, W_2$  in  $X$  whether or not  $\{W_1, W_2\}$  generates  $G$ .

The remainder of this chapter is divided into three parts. In §1.2 a brief survey of the literature concerning Problems A - D is given. This is followed in §1.3 by a summary of the main results of the thesis. §1.4 is devoted to preliminary results including the basic results of small cancellation theory. These will be used extensively in Chapter Two.

Chapter Two mainly concerns the study of Problems A - C. Some results concerning the 2GP are also obtained.

In Chapter Three Problem D and related questions are studied.

## §1.2 Survey of the literature.

In this section we give a brief survey of the literature concerning Problems A - D, in the case where the groups are given by means of a presentation. In fact very little is known about these problems in this case, since most work has been done on groups known to possess other group-theoretic properties. The question as to when a presentation defines a group having these additional properties is usually as difficult as answering the original problem.

The question of Hopficity was first raised by Hopf [1930], who, using topological methods, showed that the fundamental group of a closed two-dimensional orientable manifold is Hopfian. For a long time it was thought that all finitely generated groups would be Hopfian. However Neumann [1950] produced an infinitely related two-generator non-Hopfian group. The first finitely presented non-Hopfian group was produced by Higman [1951]. This was a three-generator, two-relator group. Many examples then followed, and Baumslag and Solitar [1962] gave a family of two-generator one-relator non-Hopfian groups.

It has been known for a long time that finitely generated (absolutely) free groups are Hopfian (see MKS, p109). A difficult open problem is whether or not finitely generated relatively free groups are Hopfian. For a discussion of this question and a summary of known results see Hanna Neumann [1967, Chapter 4].

The free product of two finitely generated Hopfian groups is again Hopfian (Dey and Neumann [1970]). It has been shown however that the free product of two Hopfian groups need not be Hopfian (M. Newman and Sichler [1973]). Also the direct product of two finitely generated Hopfian groups need not be Hopfian. In fact there exists a non-trivial finitely generated group isomorphic to its direct square (see Tyrer Jones [1974]).

Hirshon has obtained partial results concerning the question as to when the direct product of Hopfian groups is again Hopfian. For a list of this work and other references which may be of interest, see Hirshon [1975].

Finitely generated residually finite groups are Hopfian (Magnus [1969]), and many results stem from this. For example, the free product of two residually finite groups, amalgamating finite subgroups, is residually finite (Baumslag [1963]); as is the HNN extension of a residually finite group if the associated subgroups are finite (Tretkoff [1977]). The residual finiteness (and thus Hopficity) of Fuchsian groups has been known for some time. (For this result and a review of residual finiteness see Magnus [1969].) If  $G = \langle X ; R \rangle$  is Hopfian, then so is  $G = \langle X ; R^n \rangle$  (Baumslag [1967a]). Thus if  $V(X)$  and  $W(Y)$  are non-trivial freely reduced words in  $X$  and  $Y$  respectively, then  $\langle X, Y ; (VW)^n \rangle$  is Hopfian, since  $\langle X, Y ; VW \rangle$  is residually finite (Baumslag [1964]).

Dunwoody [1971] has shown that if  $|X|$  is finite  $\langle X ; R' \rangle$  is Hopfian whenever  $\langle X ; R \rangle$  is.

Pride [1977a] has shown that two-generator one-relator groups with torsion, and certain other two-generator groups, are Hopfian.

Some of Pride's results apply to small cancellation groups. Other results on the Hopficity of small cancellation groups are contained in Pride [1976b]. Also of interest are the papers of Brunner [1977], Collins [1977], Frederick [1963] and Meskin [1972], in which the Hopficity or residual finiteness of certain one-relator groups are studied.

In a series of papers, Rosenberger has used the theorem of Pride that a one-relator group with torsion which has one T-system is Hopfian, to establish the Hopficity of various one-relator groups (see Rosenberger [1977c]). In particular, he obtained some results on the Hopficity of the groups  $\langle X, Y; (UV)^n \rangle$  mentioned above.

Less is known about Problem B than is known about Problem A. Baumslag, Cannonito and Miller [1977] have shown that the automorphism group of a finitely presented group is recursively presentable. However, the automorphism group need not be finitely presentable. In fact Lewin [1967], has given an example of a 3-generator 5-relator group whose automorphism group is not finitely presentable. The paper of Frank and Kahn [1977] may also be of interest in connection with automorphism groups. Pride [1977a] has shown that the automorphism groups of two-generator one-relator groups with torsion are finitely presented. Rosenberger [1977c] has shown that certain many-generator one-relator groups with torsion have finitely generated automorphism groups.

The isomorphism problem is a difficult one. It is known that in general it is unsolvable: to be more precise there is a recursive set of finite presentations of groups such that  $ISOP(J)$  is unsolvable (Rabin [1958], Adjan [1958]). In fact there is not even a general and effective procedure to decide whether a finite



$\text{ISOP}(J)$  is solvable if all the presentations in  $J$  have no defining relations, or all the presentations in  $J$  are finite and are known to present Abelian groups (MKS p.25).

If  $J$  consists of two-generator one-relator presentations, where the relation is a proper power, then  $\text{ISOP}(J)$  is solvable (Pride [1977a]). Rosenberger [1977c] has shown  $\text{ISOP}(J)$  to be solvable when  $J$  is a recursive set of one-relator presentations satisfying certain conditions, and Meskin [1975] has solved the isomorphism problem for certain one-relator presentations.

As far as we are aware, Problem D has not been studied before. However, the results of Pride [1976a] give us a positive answer to the problem in certain cases, where  $R$  consists of a single relation and  $G$  has torsion. Further reference to Pride's results will be made in Chapter Three.

### §1.3 Survey of the thesis.

In this section we give a survey of the main results of the thesis. Many of these results concern small cancellation groups, and so we will give here a brief *resumée* of the relevant concepts concerning these groups to facilitate reading of this survey. Fuller information about these concepts is given in §1.4.

If  $R$  is a subset of a free group  $F$ , then the *symmetrized closure*  $R^*$  of  $R$  is the set consisting of all cyclically reduced conjugates of elements of  $R$  and their inverses. If  $R = R^*$  then  $R$  is said to be *symmetrized*. If  $S$  and  $T$  are distinct elements of  $R$  with  $S \equiv B\bar{S}$  and  $T \equiv B\bar{T}$ , then  $B$

is called a *piece* (relative to  $R$ ). If no element of  $R$  is the product of less than  $p$  pieces then  $R$  is said to satisfy  $C(p)$  and to be a  $C(p)$  set. A group  $G$  with presentation  $\langle X; R \rangle$ , is called a *small cancellation group* if  $R^*$  satisfies  $C(6)$ . Let  $\lambda > 0$ . If  $\ell(B) < \lambda \ell(S)$  whenever  $B$  is a piece and  $S \equiv B\bar{S} \in R$  then  $R$  is said to satisfy  $C'(\lambda)$ , and to be a  $C'(\lambda)$  set.

The main work of the thesis begins in Chapter Two.

§2.1 is elementary in nature and is concerned with obtaining information about stable subsets of the free group  $F_2$  on  $a, b$ .

Let  $R$  be a subset of  $F_2$  contained in  $\langle a \rangle^{F_2}$  but not in  $F_2'$ . Then  $R$  is said to have *simple exponents*. From our point of view the importance of sets having simple exponents stems from the fact that if  $R$  and  $S$  have simple exponents and  $\langle \phi(R) \rangle^{F_2} \subseteq \langle S \rangle^{F_2}$  for some automorphism  $\phi$  of  $F_2$ , then  $\phi$  differs by an inner automorphism from an automorphism of the form  $a \mapsto a^\epsilon, b \mapsto a^\lambda b^{\epsilon'}$  (see 2.2). This fact makes the study of stability easier by giving us an explicit form for the automorphisms of  $F_2$  which we need to consider.

It follows easily from the result just mentioned that if all elements of  $R$  are of the form

$$a^{\alpha_1} b^\epsilon a^{\alpha_2} b^{-\epsilon} \dots b^\epsilon a^{\alpha_{2n}} b^{-\epsilon} a^{\alpha_{2n+1}}$$

then  $R$  is stable (see 2.5).

A word of the above form will be called *b-alternating*. In addition a word which is the product of two *b-alternating* words (but is not itself *b-alternating*) will be called *quasi b-alternating*. We show

in 2.6 that if  $R$  is a subset of  $F_2$ , whose elements are either  $b$ -alternating or quasi  $b$ -alternating, and if  $R$  has simple exponents, then  $R$  is stable.

Also in §2.1 we show the following result. Let  $R$  have simple exponents and suppose that  $\langle \mu(R) \rangle^{F_2} = \langle R \rangle^{F_2}$  for some automorphism  $\mu$  of  $F_2$  which, up to an inner automorphism is neither the identity nor the automorphism defined by  $a \mapsto a^{-1}$ ,  $b \mapsto b^{-1}$ . Then  $R$  is stable. (See 2.7.)

In 2.8 we give an example of an unstable subset of  $F_2$  having simple exponents, for which  $\langle R \rangle^{F_2} = \langle \mu(R) \rangle^{F_2}$ , when  $\mu$  is defined by  $a \mapsto a^{-1}$ ,  $b \mapsto b^{-1}$ .

A generalisation of the idea of simple exponents is that of proportional exponents. If  $R \subseteq F_2$  and  $R \not\subseteq F_2'$ , then  $R$  is said to have *proportional exponents* if for all  $S, T \in R$ ,  $\sigma_a(S) \sigma_b(T) = \sigma_b(S) \sigma_a(T)$ . It is not difficult to show that if  $R$  has proportional exponents, then there is an automorphism  $\psi$  of  $F_2$  such that  $\psi(R)$  has simple exponents. (See 2.9.) This result, together with the fact that if  $R$  is stable then so is  $\phi(R)$  for any automorphism  $\phi$  of  $F_2$ , extends the scope of the results obtained in §2.1.

The major results of Chapter Two come in §§2.2, 2.3. In §2.2 we obtain information concerning the stability of certain small cancellation sets.

Let  $R$  be a finite set satisfying  $C'(\lambda)$ . Pride [1976b] asks if  $R$  is stable for sufficiently small  $\lambda$ . In §2.2 we give a partial answer to this question as follows. Let  $R^*$  satisfy  $C(8)$ , and have simple exponents. Suppose that the exponents of  $a$  in  $R$  are bounded in modulus. Then  $R$  is stable. (See 2.10.)

We give examples to show that, with the possible exception of the  $C(8)$  condition, none of the hypotheses of the theorem can be omitted. (See 2.11, 2.12.) These examples are, however, infinitely related.

Also in §2.2 we prove the following. *Let  $J$  be a recursive set of two-generator finite presentations. For all  $\langle a, b; R \rangle \in J$ , suppose that if  $G = \langle a, b; R \rangle$ , then  $G$  has one  $T$ -system. Moreover assume that  $R^*$  satisfies  $C'(1/8)$ , has simple exponents, and that the exponents of  $a$  in  $R$  are bounded in modulus. Then  $ISOP(J)$  is solvable.* (See 2.13.)

In §2.3 we turn to another problem raised in Pride [1976b]. Problem 3 of Pride [1976b] asks when does a small cancellation group have one  $T$ -system? Pride suggests that the case  $G = \langle X; R \rangle$ , where  $R^*$  satisfies  $C'(\lambda)$  for small enough  $\lambda$ , and where all the elements of  $R$  are proper powers, might be worth investigating. In §2.3 we concentrate on the two-generator case, and show, using the techniques of Pride [1975], that if  $G = \langle a, t; R \rangle$ , where  $R^*$  satisfies  $C'(1/24)$ , and where all elements of  $R$  are proper powers and  $t$ -alternating, then any generating pair of  $G$  is Nielsen equivalent to a pair of the form  $(a^\mu, t)$  where  $\mu$  is coprime to the order of  $a$ . Moreover  $2GP$  is solvable for  $G$ . (See 2.14.)

If  $R \not\subseteq F'(a, t)$ , it follows immediately from this and Proposition B that  $G$  has finitely generated automorphism group. Moreover, it follows from Proposition A that if  $a$  has infinite order then  $G$  is Hopfian.

Also, it follows from Proposition C that if  $J$  is the set of presentations  $\langle a, t; R \rangle$ , where  $R \not\subseteq F'$  is finite, satisfies the hypotheses of the theorem, and  $a^k \notin R$  for any  $k$  (i.e.  $a$  defines an element of infinite order), then  $ISOP(J)$  is solvable.

It turns out in fact that the last two results mentioned in the previous paragraphs still hold even when  $a$  has finite order.

The theorem stated above is just a special case of a more general result. Let  $S(a,b)$  be a symmetrized subset of  $F_2$  satisfying  $C'(1/12)$ . Suppose all elements of  $S$  are proper powers and  $a^k \in S$  if and only if  $b^k \in S$ . Let  $R(a,t) = S(a, t^{-1}at)$ . Then every generating pair of  $G = \langle a, t : R \rangle$  is Nielsen equivalent to a pair of the form  $(a^\mu, t)$  where  $\mu$  is coprime to the order of  $a$ . (See 2.15.)

This is proved in a series of lemmas. One of these, possibly of independent interest, states the following. Let  $G = \langle a, b : R \rangle$ , where  $R^*$  satisfies  $C'(1/12)$ , and where all elements of  $R$  are proper powers. Let  $W \in G$ , and  $Y = \text{sgp} \{a^{\mu_1}, W^{-1} b^{\mu_2} W\}$ . Then  $Y \cap \text{sgp} \{a\} = \text{sgp} \{a^{\mu_1}\}$ . Moreover, if  $Y \cap \text{sgp} \{b\} \neq 1$ , then  $W \stackrel{G}{=} b^\beta a^\alpha$  for some  $\alpha, \beta$ . (See 2.19.)

Chapter Three is mainly concerned with Problem D. We give examples of classes of sets  $W$  and groups  $G$  for which  $\text{sgp}_G W$  does have the obvious presentation. But first we give some examples to illustrate why  $\text{sgp}_G W$  may not have the obvious presentation. Our first positive result is : Let  $G = \langle X : R \rangle$ , and let  $W$  be a set of cyclically reduced words in  $X$  such that no  $V \in W$  is a proper power and if  $U, V \in W$  with  $U$  conjugate to  $V^{\pm 1}$  then  $U \equiv V$ . Suppose that  $W^*$  satisfies  $C(6)$  and  $R$  can be written as a set of freely reduced words  $T$  in  $W$ . Then  $\text{sgp}_G W$  has the obvious presentation. (See 3.1.)

In fact we prove more than this, but the exact result is too technical to give here.

A concept which has proved useful in studying the subgroups of groups given in terms of generators and defining relations is  $(a,b)$ -admissibility.

Pride [1976a] has proved the following. Let  $G = \langle a, X, b ; R^n \rangle$  where  $n > 1$  and let  $R$  be cyclically reduced. Let  $(U, W, V)$  be  $(a,b)$ -admissible. Then either  $\text{sgp} \{U, W, V\}$  is freely generated by  $\{U, W, V\}$  or  $\text{sgp} \{U, W, V\}$  is a one-relator group with torsion. The second possibility arises if and only if some conjugate of  $R$  can be freely expressed in the free group on  $X, a, b$ , as a word  $P(U, W, V)$  in  $U, W, V$ , in which case  $\text{sgp} \{U, W, V\}$  has presentation  $\langle f, g, h ; P^n(f, h, g) \rangle$  under the mapping  $f \mapsto U, h \mapsto W, g \mapsto V$ . (See 3.4.)

We make two generalisations of  $(a,b)$ -admissibility. The first allows us to solve Problem D for various types of presentations, and the second enables us to retain the conclusion of the above theorem.

Let  $G = \langle X ; R \rangle$ . Then  $W$  is  $n$ -admissible if  $W = (W_1, \dots, W_n)$  and

$\text{sgp}_G \{W_i, \delta(W_i) \cap \bigcup_{j=1}^{i-1} \delta(W_j)\}$  is freely generated by these for  $i = 1, \dots, n$

where  $W_0$  is taken to be the set of those elements of  $X$  not occurring in elements of  $W$ . We prove the following theorem in 3.6.

Let  $G = \langle X ; R \rangle$ , and suppose that  $W$  is  $n$ -admissible for some  $n$ . Suppose that  $R$  can be written in  $F(X)$  as freely reduced words  $T$  in  $W$ . Then  $\text{sgp } W$  has the obvious presentation.

Let  $G = \langle a, X, b ; R \rangle$ . The pair  $(U, W)$  is  $a$ -admissible (relative to  $b$ ) if all elements of  $R$  are cyclically reduced, and involve both  $a$  and  $b$ ;  $U$  is a freely reduced word in  $a, X$  which involves  $a$ ; and all elements of  $W$  are freely reduced words in  $b, X$  which freely generate a subgroup of  $G$ .

In order to generalise the result of Pride mentioned above we need, in addition to the concept of  $\alpha$ -admissibility, the following definition.

Let  $R$  be a set of cyclic words on  $X$ , and let  $W$  be a set of words  $W_i$  in  $X$  which freely generate a subgroup of  $F(X)$ . Consider the set of cyclic words in  $W$ , which when rewritten in terms of  $X$  and reduced give elements of  $R$ . Break each word  $U$  in this set to give a set of words  $T$  in  $W$ . Clearly  $T$  depends upon  $R$  and  $W$ , but we will surpress this dependence. Also  $T$  depends upon the way the words were broken, but  $T^*$  does not. In 3.7 we prove the following generalisation of Pride's result. *Let  $G = \langle a, X, b ; R^n \rangle$ ,  $n > 1$ . Let  $W = \{U, V\}$  where  $(U, V)$  is  $\alpha$ -admissible relative to  $b$ . Then  $\text{spg}_G W = \langle h ; T(h) \rangle$  under the mapping  $h \mapsto W$  and where  $T$  is defined above. In particular if  $T$  is non-empty then  $\text{sgp}_G W$  is a one-relator group with torsion.*

A similar result is obtained for certain small cancellation groups.

Unfortunately, however, we find it necessary to impose more stringent conditions on  $U$ , for the proof to work in this case. (See 3.8.)

*Let  $G = \langle a, X, b ; R \rangle$ , where  $R^*$  satisfies C(13) with all elements of  $R$  proper powers  $\geq 3$ . Let  $W = (U, V)$  be  $\alpha$ -admissible relative to  $b$ .*

*Finally, suppose that  $U$  begins and ends in  $a^\epsilon$ . Then  $\text{sgp}_G W = \langle h; T(h) \rangle$  under the mapping  $h \mapsto W$ . (see 3.3).*

An example is given to show that the condition  $\geq 3$  in the theorem cannot be replaced by  $\geq 2$ .

In the final section (§3.2) we change direction slightly and consider the malnormality of subsets of generators in certain small cancellation groups.

Recall that a subgroup  $H$  of a group  $G$  is said to be *malnormal* in  $G$  if whenever  $g \in G$ ,  $h \in H$ , and  $g^{-1}hg \in H$ , then  $g \in H$  or  $h = 1$ . B. Newman [1973] has shown that if  $G = \langle X; R^n \rangle$ , where  $n > 1$ ,  $R$  is a cyclically reduced word in  $X$ , then every subset of  $X$  generates a malnormal subgroup of  $G$ . This result has turned out to be very important in the development of the theory of one-relator groups with torsion (see for example B. Newman [1973], Pride [1975]). We generalise Newman's result as follows.

*Let  $G = \langle X, Y; R \rangle$  with  $R^*$  satisfying  $C'(1/6)$  and all elements of  $R$  which involve both  $X$ -symbols and  $Y$ -symbols are powers  $\geq 3$ . Then  $\text{sgp}_G Y$  is malnormal in  $G$ . (See 3.10.)*

An example is given to show that the condition  $\geq 3$  in the theorem cannot be replaced by  $\geq 2$ .



## §1.4 Preliminary results.

The main aim of this section is to introduce the notation and major results of small cancellation theory in a form which will be of use in later chapters. We have, where possible, used standard notation (see Schupp [1972]), but for the sake of clarity have occasionally modified it. Other useful results will also be given here for convenience.

If  $R$  is a subset of a free group  $F$  then the *symmetrized closure*  $R^*$  of  $R$  is the set consisting of all cyclically reduced conjugates of elements of  $R$  and their inverses. If  $R = R^*$ , then  $R$  is said to be *symmetrized*. If  $S$  and  $T$  are distinct elements of  $R$  with  $S \equiv \overline{BS}$  and  $T \equiv \overline{BT}$ , then  $B$  is called a *piece* (relative to  $R$ ). Where  $R$  is understood we will omit the phrase "relative to  $R$ ".

If  $S \in R$  and  $S \equiv P_1 \dots P_i T$ , where the  $P_j$ 's are all pieces, then  $T$  is called an *i-remnant* (relative to  $R$ ).

Small cancellation conditions on a set  $R$  restrict the 'size' of pieces. The most common condition is  $C'(\lambda)$  where  $\lambda$  is a non-zero positive real number.

*Condition  $C'(\lambda)$ .* If  $\ell(B) < \lambda \ell(S)$  whenever  $B$  is a piece and  $S \equiv \overline{BS} \in R$  then  $R$  is said to satisfy  $C'(\lambda)$ , and to be a  $C'(\lambda)$  set.

A more abstract condition, which does not depend upon length of words in the sense used above, is  $C(p)$  where  $p$  is a natural number.

*Condition  $C(p)$ .* If no element of  $R$  is a product of fewer than  $p$  pieces then  $R$  is said to satisfy  $C(p)$ , and to be a  $C(p)$  set.

A group  $G$  with presentation  $\langle X ; R \rangle$  is called a *small cancellation group* if  $R^*$  satisfies  $C(6)$ .

It is not difficult to see that  $C'(\lambda)$  implies  $C(q)$  for  $\lambda \leq 1/(q-1)$ , but no  $C(p)$  condition implies a  $C'(\lambda)$  condition. Both  $C'(\lambda)$  and  $C(p)$  conditions will be used in the sequel.

The fundamental result of small cancellation theory is the lemma due to Greendlinger [1960].

**1.1 GREENDLINGER'S LEMMA.** *Let  $R^*$  satisfy  $C(6)$ . Let  $W$  be a non-trivial, freely reduced consequence of  $R$ . Then for some cyclically reduced conjugate  $V$  of  $W$ , either  $V \in R^*$  or  $V$  has the form  $V \equiv \prod_{i=1}^m U_i T_i$ , where each  $T_k$  is an  $I_k$ -remnant. The number  $m$  and the numbers  $I_k$  satisfy:*

$$\sum_{1}^m (4 - I_k) \geq 6.$$

||

This lemma is the starting point of many of our proofs, though not in the form given here. We will state and prove (where necessary) some consequences of Greendlinger's lemma which are sometimes themselves called Greendlinger's lemma in the literature.

1.2 COROLLARY. *Let  $R^*$  satisfy  $C(6)$  and  $W$  be a non-trivial freely reduced consequence of  $R$ . Then for some cyclically reduced conjugate  $V$  of  $W$ , either:*

(i)  $V \in R^*$ ,

*or contains*

(ii) *two disjoint 1-remnants, or*

(iii) *three disjoint 2-remnants, or*

(iv) *four disjoint subwords, two 2-remnants and two 3-remnants, or*

(v) *five disjoint subwords, four 3-remnants and one 2-remnant, or*

(vi) *six disjoint 3-remnants.*

*In particular if  $V \notin R^*$ ,  $V$  contains  $m + 1$  disjoint  $m$ -remnants where  $0 < m \leq 3$ .*

||

The proof of this follows closely that of the Corollary of Theorem I of Schupp [1970], and is therefore omitted.

We will need some information on the way pieces behave, and so we define the following function (the 'piece' function)  $p$  from the set of freely reduced words in  $X$  to the extended natural numbers.

$p(W)$  = minimum number of pieces in which  $W$  can be written, or  $\infty$ .

$p$  will be useful when dealing with  $C(p)$  sets and corresponds with  $\ell$  when dealing with  $C'(\lambda)$  sets.

1.3 LEMMA. *If  $UV$  is freely reduced, then*

$$p(U) + p(V) - 1 \leq p(UV) \leq p(U) + p(V).$$

*Proof.* If one of  $p(U)$  or  $p(V) = \infty$ , the lemma holds. If

$U \equiv P_1 \dots P_r$ ,  $V \equiv Q_1 \dots Q_s$  with  $P_i, Q_i$  pieces then

$UV \equiv P_1 \dots P_r Q_1 \dots Q_s$  implies that

$$p(UV) \leq p(U) + p(V).$$

If  $UV \equiv W_1 \dots W_i W_{i+1} \dots W_t$ , with  $W_i$  pieces, then for some  $1 \leq i \leq t$

$$U \equiv W_1 \dots W_{i-1} W_i' \text{ and } V \equiv W_i'' W_{i+1} \dots W_t.$$

with  $W_i \equiv W_i' W_i''$ , and since a subword of a piece is a piece

$$1 + p(UV) \geq p(U) + p(V).$$

||

1.4 COROLLARY. (1) *If  $W \equiv S_1 \dots S_n$  is freely reduced, then*

$$p(S_1) + \dots + p(S_n) + 1 - n \leq p(W) \leq p(S_1) + \dots + p(S_n).$$

(2) *If  $Z$  is cyclically reduced,*

$$\frac{p(Z^n)}{n} \leq p(Z) \leq \frac{p(Z^n) - 1}{n} + 1$$

and

$$n(p(Z) - 1) + 1 \leq p(Z^n) \leq n p(Z).$$

||

1.5 LEMMA. Let  $R^*$  satisfy  $C(6)$  and let  $W$  be a non-trivial cyclically reduced consequence of  $R$ . Suppose also that  $W \notin R^*$ . Then some cyclically reduced conjugate  $V$  of  $W$  contains a 3-remnant,  $T$ , of some  $S \in R^*$  with  $2 + p(S) < p(V)$ .

Proof. Some cyclically reduced conjugate  $V$  of  $W$  contains  $m + 1$  disjoint  $m$ -remnants  $T_i$  of  $S_i \in R^*$ . Without loss of generality we may assume that

$$V \equiv \prod_{i=1}^{m+1} U_i T_i, \quad 1 \leq m \leq 3$$

So

$$p(V) \geq \sum_1^{m+1} p(U_i T_i) - m \text{ by 1.4}$$

$$\geq \sum_1^{m+1} p(T_i) - m.$$

Let  $S$  be one of the  $S_i$ 's with minimum  $p(S_i)$ ,  $p(\text{say})$ , then

$p(V) \geq (m + 1)(p - m) - m$  and if  $2 + p(S) \geq p(V)$  then

$2 + p \geq p(V) \geq (m + 1)(p - m) - m$  so that  $(m + 2)m + 2 \geq m p$  :

in other words  $p \leq (m + 2) + \frac{2}{m} < 6$ , contradicting the  $C(6)$

condition.

||

1.6 COROLLARY. Let  $R^*$  satisfy  $C'(\frac{1}{6})$  and let  $W$  be a non-trivial cyclically reduced consequence of  $R$ . Suppose also that  $W \notin R^*$ . Then some cyclically reduced conjugate  $V$  of  $W$  contains a 3-remnant,  $T$ , of some  $S \in R^*$  with  $\ell(S) < \ell(V)$ .

||

1.7 LEMMA. Suppose  $a^\alpha$ ,  $\alpha > 1$  occurs in  $S \in R^*$ . Then either

1.  $S \equiv a^\gamma$  for some  $\gamma \geq \alpha$ , or
2.  $p(a^\alpha) = 1$ , or
3.  $p(a^\alpha) = 2$  and  $S \sim (a^\alpha T)^n$

where  $T$  starts and ends in symbols other than  $a^{\pm 1}$ ,  $a^\alpha T$  is not a proper power and  $T$  does not contain  $a$  with exponent  $\beta$ , where  $|\beta| \geq \alpha$ .

Proof. Suppose neither 1 nor 2 hold. We will show that 3 holds.

Case (i) Suppose  $a^\gamma U \in R^*$  for some  $U$  where  $\gamma > \alpha$ . Then  $a^\alpha a^{\gamma-\alpha} U$  and  $a^\alpha U a^{\gamma-\alpha} \in R^*$ . So  $a^{\gamma-\alpha} U \equiv U a^{\gamma-\alpha}$  if  $a^\alpha$  is not a piece, but  $a^{\gamma-\alpha}$  and  $U$  commute if and only if they are powers of a common element, so both are powers of  $a$  and so 1 or 2 holds. So  $a$  does not occur in any element of  $R^*$  with exponent  $\beta$ , where  $|\beta| > \alpha$ .

Case (ii) Suppose  $a^\alpha S a^{\epsilon\alpha} T \in R^*$  where  $S$  does not contain  $a^{\pm\alpha}$ .

We will first show that  $\epsilon = 1$ . For otherwise  $a^\alpha S a^{-\alpha} T$  and  $a^\alpha S^{-1} a^{-\alpha} T^{-1} \in R^*$  making  $a^\alpha$  a piece, i.e.  $p(a^\alpha) = 1$ . (So 2 holds contrary to assumption.)

Now  $a^\alpha S a^\alpha T$  and  $a^\alpha T a^\alpha S \in R^*$ , without loss of generality both  $S$  and  $T$  begin and end in a non  $a$ -symbol for otherwise (i) above holds, so  $a^\alpha$  is a piece unless  $S a^\alpha T \equiv T a^\alpha S$ . So  $a^\alpha S a^\alpha T \equiv a^\alpha T a^\alpha S$  and  $a^\alpha S, a^\alpha T$  commute and so are powers of a common element. Clearly  $a^\alpha S \equiv W^n$  implies either that  $W \equiv a^\mu$  or that  $W \equiv a^\alpha V$  for some initial segment  $V$  of  $S$ . So  $(a^\alpha V)^m \in R^*$ . Also  $a^{\alpha-1}$  is a piece by the above, and so  $p(a^\alpha) \leq 2$ , and the lemma follows.

||

The following consequences of the  $C(p)$  and  $C'(\lambda)$  conditions will be needed.

1.8 REMARKS. Let  $G = \langle X ; R \rangle$ , where  $R^*$  satisfies  $C(6)$ , then:

- (1) If  $x$  is a piece and  $x^\alpha W \in \langle R \rangle^F$ , then either  $W$  is a power of  $x$  in  $F(X)$  or  $W$ , when reduced contains a 3-remnant of an element  $S \in R^*$ .

To see this, note that  $x^\alpha W$  may be assumed to be cyclically reduced, and so by 1.2 either  $x^\alpha W \in R^*$  or contains  $m+1$  disjoint  $m$ -remnants of elements  $S_i \in R^*$  (when written on a circle). Using 1.7 the results follow.

(2) If  $a^\alpha$  occurs in  $S \in R^*$  with  $S \neq a^\beta$ , then  
 $|\alpha| < 1 + \lambda \ell(S)$  if  $R^*$  satisfies  $C'(\lambda)$ .

(3) If  $a$  and  $b$  are pieces of length 1 and  $b$  is not  $a$  or  $a^{-1}$   
 then  $a^\alpha b^\beta = 1$  if and only if  $a^\alpha = 1$  and  $b^\beta = 1$ .

||

The following result of B. Newman as extended by Gurevič<sup>v</sup> will  
 be of use.

1.9 LEMMA. (The Newman-Gurevič<sup>v</sup> spelling theorem).

Let  $G = \langle X ; P^m \rangle$  where  $P$  is cyclically reduced and  $m > 1$ . Suppose  
 $W = V$  in  $G$ , where  $W$  is a freely reduced word, and  $V$  omits a generator  
 $y$  of  $G$  which occurs in both  $W$  and  $P$ . Then  $W$  has a subword  $(y^\epsilon S)^{m-1} y^\epsilon$ ,  
 where  $y^\epsilon S$  is a cyclic permutation of  $P^{\pm 1}$ .

This result has been strengthened further by Schupp. (See B. Newman  
 [1968], Gurevič [1972], Schupp [1976].)

||

The next lemma is of great use in Chapter Two. It is rather  
 technical in nature but its proof is quite elementary.



1.10 LEMMA. Let  $R$  be a subset of  $F(X)$  and  $U_1, U_2$  words in  $X$  with  $p(U_i) \geq 2$  (relative to  $R^*$ ). Suppose that  $U_i$  are subwords of  $V$  which is itself a subword of words  $W, S$  with  $S \in R^*$ . Finally suppose that  $U_i$  is also a subword of  $T_i \in R^*$ . Then the words obtained from  $W$  by replacing  $U_1$  or  $U_2$  in it by their complements in  $T_1$  or  $T_2$  and freely reducing are the same.

Proof. We have (say)

$$S \equiv S_1 V S_2 \in R^*,$$

$$W \equiv W_1 V W_2,$$

$$V \equiv V_1 U_1 V_2 \equiv V_3 U_2 V_4,$$

and  $T_i \equiv \overline{T_i} U_i \overline{T_i} \in R^*.$

We need to show that  $W_1 V_1 \overline{T_1}^{-1} \overline{T_1}^{-1} V_2 W_2 \stackrel{F}{=} W_1 V_3 \overline{T_2}^{-1} \overline{T_2}^{-1} V_4 W_2$

This is true if  $V_1 \overline{T_1}^{-1} \overline{T_1}^{-1} V_2 \stackrel{F}{=} V_3 \overline{T_2}^{-1} \overline{T_2}^{-1} V_4.$

Now since  $U_1$  is not a piece and

$$U_1 \overline{T_1} \overline{T_1}, U_1 V_2 S_2 S_1 V_1 \in R^*$$

then

$$\overline{T_1} \overline{T_1} \equiv V_2 S_2 S_1 V_1.$$

Similarly

$$\overline{T_2} \overline{T_2} \equiv V_4 S_2 S_1 V_3.$$

So we need to show that

$$V_1 V_1^{-1} S_1^{-1} S_2^{-1} V_2^{-1} V_2 \stackrel{F}{=} V_3 V_3^{-1} S_1^{-1} S_2^{-1} V_4^{-1} V_4,$$

which is clearly the case.

||

In fact similar results are true about replacing subwords by their complements and when these are used in the rest of the thesis reference will be made to 1.10. The proofs of these generalisations are similar to the proof of 1.10.

1.11 REMARKS. 1. Let  $R$  be cyclically reduced word in  $X$  with  $R=Z^n$  then  $Z$  is not a piece relative to  $R^*$ .

2.If  $S \cup R$  satisfies  $C(6)$  then  $S \notin \langle R \rangle^F$  or  $S \in R^*$

3.If  $S, T \in R$  both contain  $B$  as a subword and  $p(B) > 1$ , then  $S \sim T$ .

||

We need an idea similar to that of a symmetrized set, which will be of use in Chapter Three. Let  $R$  be a set of freely reduced words in  $X$ . Denote by  $R_*$  the smallest set of words in  $X$  which contains  $R$  and is closed under taking inverses and cyclic permutations. The elements of  $R_*$  need not be freely reduced. We may, of course define a piece relative to  $R_*$  in the natural way.

## CHAPTER TWO

The main results of this chapter are contained in §§2.2, 2.3.

In §2.1, we study stability in general and the stability of two-generator sets in particular. We go on to the detailed study of the stability of certain two-generator small cancellation groups in §2.2. Then follows a consideration of the T-systems of certain small cancellation groups, which constitutes §2.3.

### §2.1 Elementary results on stability.

In this section we will obtain information concerning stable subsets of the free group of rank 2. Before doing this, however, we will make some general remarks. Let  $F$  be a free group and  $R \subseteq F$ . Then  $R$  is said to be *unstable under*  $\phi$ ,  $\phi \in \text{Aut}(F)$  if  $\langle \phi(R) \rangle^F \subsetneq \langle R \rangle^F$ . Clearly  $R$  is unstable if and only if it is unstable under some  $\phi \in \text{Aut}(F)$ .

2.1 REMARKS. Suppose  $R$  is unstable under  $\phi$ . Then

1.  $\phi^m(R)$  is unstable under  $\phi^n$  for  $n > 0$ .
2.  $\langle \phi^k(R) \rangle^F = \langle \phi^l(R) \rangle^F$  if and only if  $l = k$ .
3. If  $\psi \in \text{Aut}(F)$  then  $\psi(R)$  is unstable under  $\psi \phi \psi^{-1}$ .

The above results are trivial consequences of the definition of unstable.

||

From now on  $F_2$  will denote the free group freely generated by  $\{a, b\}$ . Let  $R$  be a subset of  $F_2$  contained in  $\langle a \rangle^{F_2}$  but not in  $F_2'$ . Then  $R$  is said to have *simple exponents*. The name derives from the fact that if  $R$  has simple exponents, then the exponent sum of  $b$  in every element of  $R$  is zero and the exponent sum of  $a$  in some element of  $R$  is non-zero.

Let  $G = \langle a, b; R \rangle$  with  $R$  having simple exponents. Then either each element of  $R^*$  is of the form  $a^\alpha$  or both  $a$  and  $b$  are pieces. To see this suppose that  $S \in R^*$  is not of the form  $a^\alpha$ . Then it is easy to see that there are cyclic permutations  $S_1$  and  $S_2$  of  $S$  with

$$S_1 \equiv Aba^{\varepsilon_1 \alpha} b^{-1}$$

$$S_2 \equiv Bb^{-1} a^{\varepsilon_2 \beta} b$$

$|\varepsilon_i| = 1$ , and  $\alpha, \beta > 0$ . We do not claim that  $A \neq B$ . From this we see that the following four words belong to  $R^*$ :

$$Aba^{\varepsilon_1 \alpha} b^{-1}, A^{-1}ba^{-\varepsilon_1 \alpha} b^{-1}$$

$$Bb^{-1}a^{\varepsilon_2 \beta} b, B^{-1}b^{-1}a^{-\varepsilon_2 \beta} b$$

That  $b$  is a piece can be seen by comparing the first two words. That  $a$  is a piece follows from the fact that either  $\varepsilon_2 \beta$  or  $-\varepsilon_2 \beta$  has the same sign as  $\varepsilon_1 \alpha$ .

**2.2 LEMMA.** Suppose  $R$  and  $S$  have simple exponents and  $\langle \phi(R) \rangle^{F_2} \subseteq \langle S \rangle^{F_2}$  for some automorphism  $\phi$  of  $F_2$ . Then  $\phi$  differs by an inner automorphism from an automorphism  $a \mapsto a^\varepsilon, b \mapsto a^\lambda b^{\varepsilon'}$ ; ( $|\varepsilon| = |\varepsilon'| = 1$ .)

The proof of this stems from the fact that the kernel of the natural homomorphism from  $\text{Aut}(F_2)$  onto  $\text{Aut}(F_2/F_2')$  is precisely the group of inner automorphisms [MKS, p.169]. The elements of  $\text{Aut}(F_2/F_2')$  have the form

$$a \mapsto a^\alpha b^\beta; \quad b \mapsto a^\gamma b^\delta$$

with  $|\alpha\delta - \beta\gamma| = 1$ .

Since the exponent sum of  $b$  is zero in all elements of  $S$ , it must be zero in all  $\phi(T), T \in R$ . If  $T$  is an element of  $R$  with  $\sigma_\alpha(T) \neq 0$  then this can only happen if  $\beta=0$  and  $\alpha\delta = \pm 1$ .

||

The case  $R = S$  will be of use in this section and  $R \neq S$  in §2.2.

The critical part of the proof of the above lemma was that the kernel of the natural homomorphism from  $\text{Aut}(F_2)$  onto  $\text{Aut}(F_2/F_2')$  is precisely the inner automorphism group. This is no longer true for  $F_3$  (see MKS p.168). In this regard it is worth pointing out that Higman [1951] has shown that there is a subset of  $F_3$  unstable under an automorphism contained in the kernel of the natural map from  $\text{Aut}(F_3)$  onto  $\text{Aut}(F_3/F_3')$ . Also, 2.2 does not hold if the condition that  $R \not\subseteq F'$  is dropped, since  $\langle \phi([a,b]) \rangle^{F_2} \subseteq \langle [a,b] \rangle^{F_2}$  for any  $\phi \in \text{Aut}(F_2)$ .

Let  $\eta$ ,  $\chi_a$  and  $\chi_b$  denote the automorphisms of  $F_2$  given by:

$$\eta : \begin{array}{l} a \mapsto a \\ b \mapsto ab \end{array}; \quad \chi_a : \begin{array}{l} a \mapsto a^{-1} \\ b \mapsto b \end{array}; \quad \chi_b : \begin{array}{l} a \mapsto a \\ b \mapsto b^{-1} \end{array}$$

2.3 REMARK. The following relations between these automorphisms are easily checked:

$$\chi_a \chi_b = \chi_b \chi_a;$$

$$\chi_a^2 = \chi_b^2 = 1;$$

and up to an inner automorphism

$$\chi_a \eta = \eta^{-1} \chi_a; \quad \chi_b \eta = \eta^{-1} \chi_b.$$

We now note some consequences of 2.2.

2.4 LEMMA. If  $R$  has simple exponents and  $\langle \phi(R) \rangle^{F_2} \subseteq \langle R \rangle^{F_2}$  for some automorphism  $\phi$  of  $F_2$ , then  $\phi$  is equal (up to an inner automorphism), to ~~one of~~

$$\eta^\lambda, \chi_a \eta^\lambda, \chi_b \eta^\lambda \text{ or } \chi_a \chi_b \eta^\lambda.$$

If strict containment holds then  $R$  is unstable under  $\phi$  and is unstable under  $\eta^\alpha$  for some  $\alpha$  which can be chosen so that  $|\alpha|$  is as large as we like.

The first part of the lemma follows from 2.2 and the definitions of  $\eta$ ,  $\chi_a$  and  $\chi_b$ . The rest of the lemma then follows from 2.1 and 2.3.

2.5 REMARK. Now let  $G = \langle a, b; R \rangle$ . It follows immediately from 2.2 that if  $a$  has finite order in  $G$ , and  $R$  has simple exponents, then  $R$  is stable. Also if all elements of  $R$  are of the form:

$$(*) \quad a^{\alpha_1} b^{\epsilon_1} a^{\alpha_2} b^{-\epsilon_2} \dots a^{\alpha_{2n}} b^{-\epsilon_n} a^{\alpha_{2n+1}}$$

then  $R$  is stable.

This follows from 2.4 and the fact that  $\langle \eta^\lambda(R) \rangle^{F_2} = \langle R \rangle^{F_2}$  for all  $\lambda$  (see Pride [1976b]). A word of the form(\*) will be called *b-alternating*.

We can generalise this latter result as follows: a word in  $a$  and  $b$  will be said to be *quasi b-alternating* if it is not *b-alternating*, involves  $b$  and is the product of two *b-alternating* words. Thus up to inverses and cyclic permutations, quasi *b-alternating* words have the form:

$$(\prod_i a^{\alpha_i} b^{\beta_i} a^{\gamma_i} b^{-1}) (\prod_j a^{\gamma_j} b^{-1} a^{\delta_j} b)$$

2.6 THEOREM. Let  $R_1 \subseteq F_2$ , be a set of quasi *b-alternating* words, and  $R_2 \subseteq F_2$  be such that  $\langle R_2 \rangle^{F_2}$  is fixed by  $\eta^\gamma$  for some  $\gamma > 0$ . Let  $R = R_1 \cup R_2$  and suppose that  $R$  has simple exponents. Then  $R$  is stable.

Proof. By way of contradiction suppose  $R$  is unstable. Then by 2.4  $R$  is unstable under  $\eta^\lambda$  for some  $\lambda$ . By 2.1  $R$  is unstable under  $\eta^{\lambda\gamma}$ .

We will show that

$$\langle \eta^{-\lambda\gamma}(R) \rangle^{F_2} \supset \langle R \rangle^{F_2} \supseteq \langle \eta^{-\lambda\gamma}(R) \rangle^{F_2},$$

which is clearly contradictory.

The first containment holds by the definition of stability. Let  $G = \langle a, b; R \rangle$  and  $S \in R_1$ . Since  $S$  is quasi  $b$ -alternating  $S^E \sim UV$ , where

$$U \equiv \prod_i a^{\alpha_i} b a^{\beta_i} b^{-1} \text{ and } V \equiv \prod_j a^{\gamma_j} b^{-1} a^{\delta_j} b.$$

Now  $\eta^{\lambda\gamma}(S^E) \sim \eta^{\lambda\gamma}(U)\eta^{\lambda\gamma}(V) = a^{\lambda\gamma} U a^{-\lambda\gamma} V$ , so  $a^{\lambda\gamma} U a^{-\lambda\gamma} V \stackrel{G}{=} 1$ , since

$$\langle \eta^{\lambda\gamma}(R) \rangle^{F_2} \subset \langle R \rangle^{F_2}. \text{ Thus } a^{\lambda\gamma} U a^{-\lambda\gamma} U^{-1} = 1 \text{ since } U = V^{-1}. \text{ So}$$

$$[a^{\lambda\gamma}, U] = 1. \text{ Similarly}$$

$$\eta^{-\lambda\gamma}(S^E) \sim \eta^{-\lambda\gamma}(U)\eta^{-\lambda\gamma}(V) = a^{-\lambda\gamma} U a^{\lambda\gamma} V.$$

But  $a^{-\lambda\gamma} U a^{\lambda\gamma} V = a^{-\lambda\gamma} U a^{\lambda\gamma} U^{-1} = 1$  since  $[a^{\lambda\gamma}, U] = 1$ . So we have shown that  $\langle R \rangle^{F_2} \supseteq \langle \eta^{-\lambda\gamma}(R_1) \rangle^{F_2}$  and since  $\langle \eta^{-\lambda\gamma}(R_2) \rangle^{F_2} = \langle R_2 \rangle^{F_2}$ . We have  $\langle R \rangle^{F_2} \supseteq \langle \eta^{-\lambda\gamma}(R) \rangle^{F_2}$ , as required.

||

It is clear from 2.6 that if  $R$  has simple exponents and  $\langle R \rangle^{F_2}$  is fixed by  $\eta^\lambda$  for some  $\lambda \neq 0$  then  $R$  is stable. More generally we have:

**2.7 THEOREM.** *Let  $R$  have simple exponents and suppose that  $\langle \mu(R) \rangle^{F_2} = \langle R \rangle^{F_2}$  for some automorphism  $\mu$  of  $F_2$  which, up to an inner automorphism, is neither  $\chi_a \chi_b$  nor the identity. Then  $R$  is stable.*

**Proof.** By way of contradiction assume that  $R$  is unstable, and so by 2.4  $R$  is unstable under  $\eta^\lambda$  for some  $\lambda$ . Again by 2.4, up to an inner automorphism  $\mu$  is one of  $\eta^\alpha$ ,  $\chi_a \eta^\alpha$ ,  $\chi_b \eta^\alpha$  or  $\chi_a \chi_b \eta^\alpha$  for suitable  $\alpha$ . The case  $\mu = \eta^\alpha$  follows from 2.6 as does the case that  $\mu = \chi_a \chi_b \eta^\alpha$ , since  $\mu^2 = r^{2\alpha}$  up to an inner automorphism. We need only consider  $\mu = \chi_a \eta^\alpha$  as a similar argument applies in case

$\mu = \chi_b \eta^\alpha$ . We have  $\langle k \rangle^{F_2} = \langle \chi_\alpha \eta^\alpha(R) \rangle^{F_2}$  and  $\langle \eta^{-\lambda}(R) \rangle^{F_2} \supset \langle R \rangle^{F_2} \supset \langle \eta^\lambda(R) \rangle^{F_2}$ .

But

$$\begin{aligned} \langle \eta^\lambda(R) \rangle^{F_2} &= \langle \eta^\lambda \chi_\alpha \eta^\alpha(R) \rangle^{F_2} = \langle \chi_\alpha \eta^{-\lambda} \eta^\alpha(R) \rangle^{F_2} \quad , \text{by 2.3} \\ &= \langle \chi_\alpha \eta^{\alpha-\lambda}(R) \rangle^{F_2} \end{aligned}$$

so

$$\langle R \rangle^{F_2} = \langle (\chi_\alpha \eta^\alpha)^{-1}(R) \rangle^{F_2} \supset \langle \eta^{-\lambda}(R) \rangle^{F_2}.$$

Thus

$$\langle \eta^{-\lambda}(R) \rangle^{F_2} \supset \langle R \rangle^{F_2} \supset \langle \eta^{-\lambda}(R) \rangle^{F_2},$$

which is a clear contradiction.

||

The following example shows that the condition  $\mu \neq \chi_a \chi_b$  cannot be relaxed.

2.8 EXAMPLE. Let  $S_i = ab^{-1}a^{-i}b^{-1}aba^i b$  and  $R_j = \{S_j^n, S_{j+1}^n, \dots\}$ .

Notice that  $R_j$  has simple exponents. Since  $S_i^{\pm 1}$  on a circle is not a subword of  $S_k$  for  $i \neq k$ ,  $R_j^*$  satisfies  $C'(1/n)$ . So if  $n \geq 6$ ,  $S_j^n \notin \langle S_{j+1}^n, \dots \rangle^{F_n}$  by 1.11. Thus  $\langle R_j \rangle^{F_2} \not\supset \langle R_k \rangle^{F_2}$  if  $j < k$ .

Now  $\eta S_i = ab^{-1}a^{-1}a^{-i}b^{-1}a^{-1}aaba^i ab = S_{i+1}$  so that  $R_k = \eta^{k-j} R_j$  for  $j < k$ .

Thus  $\langle R_j \rangle^{F_2} \supset \langle \eta^{k-j} R_j \rangle^{F_2}$  and so  $R_j$  is unstable under  $\eta^{k-j}$ . Finally note that  $\chi_a \chi_b(S_i) = a^{-1}ba^i ba^{-1}b^{-1}a^{-i}b^{-1} \sim S_i^{-1}$ , so  $R_j$  satisfies 2.7 except for the condition  $\mu \neq \chi_a \chi_b$  (up to an inner automorphism).

It is interesting that in the above example if  $n=1$  then 2.6 applies and  $R_j$  is stable.

A generalisation of the idea of simple exponents for which the results of this section are applicable is that of proportional exponents. If  $R \in F_2$  and  $R \not\in F_2'$  then  $R$  is said to have *proportional exponents* if for all  $S, T \in R$ ,  $\sigma_a(S)\sigma_b(T) = \sigma_a(T)\sigma_b(S)$ . Clearly, if  $R$  has simple exponents it has proportional exponents.



The next elementary but useful result gives the relationship between sets with simple and sets with proportional exponents.

2.9 LEMMA. *Let  $R$  have proportional exponents. Then there is an automorphism  $\psi$  of  $F_2$  such that  $\psi(R)$  has simple exponents.*

Proof. Not all  $S \in R$  have both  $\sigma_a(S) = 0$  and  $\sigma_b(S) = 0$ , otherwise  $R \subseteq F'$ . If  $\sigma_b(S) \neq 0$  but  $\sigma_a(S) = 0$  for some  $S \in R$  then  $a \mapsto b, b \mapsto a$  can easily be seen to be the required automorphism. So the only case left is if for some  $S \in R$ ,  $\sigma_a(S)\sigma_b(S) \neq 0$ . By applying  $\chi_a, \chi_b$  and interchanging  $a$  and  $b$  if necessary we may assume that  $\sigma_a(S) \geq \sigma_b(S) > 0$ . Applying  $a \mapsto a, b \mapsto a^{-1}b$  yields  $S'$  with  $0 \leq \sigma_a(S') = \sigma_a(S) - \sigma_b(S) < \sigma_a(S)$  and  $\sigma_b(S') = \sigma_b(S) \neq 0$ . Continuing this process we obtain the required result.

||

The generalisation of the results of this section to  $R$  having proportional exponents now follows from 2.1, since if  $R$  is unstable, then so is  $\psi(R)$ , and  $\psi(R)$  has simple exponents. Thus, let  $R$  have proportional exponents and either:

(i)  $\langle \mu(R) \rangle^{F_2} = \langle R \rangle^{F_2}$  for some  $\mu \in \text{Aut } F_2$  which is not (up to an inner automorphism) the identity nor  $\chi_a \chi_b$ ;

or

(ii)  $\langle R \rangle^{F_2}$  contains the non-zero power of a primitive:  
then  $R$  is stable.

The truth of the results of this section depend heavily on  $F_2$  being two-generator. For the set  $R = \{a^{-1}cac^{-2}, b^{-1}cbc^{-2}\}$  of Higman [1951] satisfies:

- (i)  $\sigma_a(S) = \sigma_b(S) = 0 \neq \sigma_c(S)$  for all  $S \in R$ ;
- (ii)  $\langle \mu(R) \rangle^{F_3} = \langle R \rangle^{F_3}$ , where  $\mu : a \mapsto b, b \mapsto a, c \mapsto c$ . But  $R$  is unstable under the automorphism defined by

$$a \mapsto a, b \mapsto b, c \mapsto a^{-1}ca.$$

## §2.2 The stability of certain small cancellation sets.

Let  $R$  be a finite set satisfying  $C'(\lambda)$ . Pride [1976b] asks if  $R$  is stable for sufficiently small  $\lambda$ . In this section we will give a positive answer to this question in certain cases. In fact we prove:

**2.10 THEOREM.** *Let  $R^*$  satisfy  $C(8)$ , have simple exponents and suppose that the exponents of  $a$  in  $R$  are bounded in modulus by  $k > 0$ . Then  $R$  is stable.*

Since  $C'(1/7)$  sets satisfy  $C(8)$ , the theorem remains true if the  $C(8)$  condition is replaced by the  $C'(1/7)$  condition.

Before going on to prove this theorem, we will give examples to show that the hypothesis that  $R$  has simple exponents, and that the  $a$ -exponents in  $R$  are bounded, cannot in general be dispensed with. First note that 2.8 with  $n \geq 8$  gives an example of an unstable subset of  $F_2$  satisfying  $C(8)$  and having simple exponents but in which the  $a$ -exponents are not bounded. Next we give an example of an unstable subset of  $F_2$  satisfying  $C(8)$  and having bounded, but not simple, exponents.

**2.11 EXAMPLE.** Let  $S_i = a(ab)^i$ ,  $i \geq 1$ ,  $R_j = \{S_j^n, S_{j+1}^n, \dots\}$ . Then the exponents in  $R_j$  are all bounded and since no subword of the form  $a(ab)^i a^2$  is a piece,  $R_j^*$  satisfies  $C'(2/n)$  whenever  $n > 12$ , say. So  $R_j$  can be made to satisfy any  $C(p)$  condition. If  $\phi : a \mapsto ba^2, b \mapsto a^{-1}$ ,

then  $\phi(S_i) = ba^2(ba^2a^{-1})^i = ba^2(ba)^i \sim S_{i+1}$ , clearly  $R_j$  is unstable under  $\phi$  when  $n > 12$ .

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Finally, we give an example of an unstable subset  $R$  of  $F_2$  satisfying  $C(8)$ , having bounded exponents with  $R \subseteq \langle a \rangle^{F_2}$  but with  $R \not\subseteq F_2'$ .

2.12 EXAMPLE. Let  $S_i = a(b^{-1}a^{-1})^i b^{-2}a^{-1}(ba)^i b^2$  and  $R_j = \{S_j^n, S_{j+1}^n, \dots\}$   $n > 6$ . Since  $S_i$  is not a subword of  $S_k$  if  $i \neq k$  (when written on a circle),  $R_j$  satisfies  $C'(1/n)$ . Also if  $\psi : a \mapsto ba, b \mapsto b$ , then  $\psi(S_i) \sim S_{i+1}$ .  $R_j$  is thus unstable under  $\psi$ . However, the theorem does not apply, since  $R_j \subseteq F_2'$ .

||

If  $R$  is as in the theorem, and  $G = \langle a, b; R \rangle$  has one T-system, then  $G$  is Hopfian by Proposition A. It will be shown in the next section that various two-generator small cancellation groups have one T-system, but it should be mentioned that for these groups the relators are all  $b$ -alternating, so the stability follows immediately from Pride [1976b]. Nevertheless, it is probably true that many groups to which 2.10 applies will turn out to have one T-system.

By making use of 2.10, the following theorem will also be proved in this section.

2.13 THEOREM. Let  $J$  be a recursive set of two-generator finite presentations. For every  $\langle a, b; R \rangle \in J$ , suppose that if  $G = \langle a, b; R \rangle$ , then  $G$  has one T-system. Moreover assume  $R^*$  satisfies  $C'(1/8)$ , has simple exponents, and the exponents of  $a$  in  $R$  are bounded in modulus by  $k$ .

Then ISOP(J) is solvable. We now go on to prove 2.10 and 2.13.

**Proof of 2.10.** If  $a$  has finite order in  $G$  or all elements of  $R$  are  $b$ -alternating, the theorem follows from 2.5. In the following argument the reader is reminded that in the case that  $a$  has infinite order, we have shown in the discussion following 2.1 that both  $a$  and  $b$  are pieces. By way of contradiction

let  $R$  be unstable under  $\phi$ . By 2.4,  $\phi$  can be taken to be  $\eta^\lambda$  where  $|\lambda| > 2k$ . Let  $S$  be a non  $b$ -alternating element of  $R$ . By taking a cyclic permutation of  $S^{\pm 1}$  if necessary, we may assume that  $S \equiv S_1 a^{\alpha_1} S_2 \dots S_n a^{\alpha_n}$ , where  $S_i$  begins in  $b^{\epsilon_i}$  and ends in  $b^{\delta_i}$ , the exponents of  $b$  in  $S_i$  alternate between  $\pm 1$ , and  $|\delta_i| = |\epsilon_i| = -\epsilon_1 = 1$  and  $\epsilon_i = \delta_{i-1}$ . Clearly  $n > 1$ . Now  $\phi(S) = S_1 a^{1+\epsilon_2 \lambda} S_2 \dots a^{n-1+\epsilon_n \lambda} S_n a^{n-\lambda}$ . Notice that if  $a^\gamma B a^\gamma$  is a  $b$ -alternating subword of  $\phi(S)$  then  $a^\gamma B$  or  $B a^\gamma$  is a subword of  $S$ . Now  $\phi(S) \in \langle R \rangle^{F_2}$  and is cyclically reduced. The condition on  $\lambda$  and the fact that  $S$  is not  $b$ -alternating ensures that  $\phi(S) \notin R^*$ . By 1.2  $\phi(S)$ , when written on a circle must contain  $m+1$  ( $m=1, 2, 3$ ) disjoint  $m$ -remnants  $T_i$  of elements  $R_i \in R^*$ . In particular,  $p(T_i) \geq 5$  for each  $T_i$ , since  $p(R_i) \geq 8$ . Now, again because of the choice of  $\lambda$ , the  $T_i$  must be subwords of  $a^{\alpha_{j-1}+\epsilon_j \lambda} S_j a^{\alpha_j+\delta_j \lambda}$  (where  $\alpha_0 = \alpha_n$ ). So  $T_i \equiv a^{\gamma_i} B_i a^{\gamma_i'}$  where  $B_i$  begin and end in  $b$ -symbols, are disjoint subwords of  $S$ , when written on a circle, and do not contain a subword of the form  $\dots b^{\epsilon} a^{\delta} b^{\epsilon} \dots$ .

Suppose first that no  $B_i$  is a piece, so that  $p(B_i) \geq 2$ . Since  $B_i$  occurs in both  $R_i$  and  $S$ ,  $R_i \sim S$  by 1.10. So  $T_1$  is an  $m$ -remnant of  $R_1 \equiv T_1 C$  with the other  $B_i, i \neq 1$  disjoint subwords of  $C$  and  $p(C) \leq m$ . But by 1.3, and 1.4

$$p(C) \geq \sum_2^{m+1} p(B_i) - m + 1 \geq 2m - m + 1 = m + 1 \text{ clearly } m \geq m + 1 \text{ is a contradiction.}$$

Thus at least one of the  $B_i$ 's, say  $B_1$ , is a piece. Now

$$p(T_1) \leq p(a^{\gamma_1}) + p(B_1) + p(a^{\gamma_1'}), \leq 2 + 1 + 1 = 4 < 5 \text{ unless both } p(a^{\gamma_1}) \text{ and } p(a^{\gamma_1'}) \geq 2. \text{ By 1.7 this can only happen if } \gamma_1 = \gamma_2, R_1 \sim (a^{\gamma_1} B)^n \text{ and}$$

$B_1 \equiv U(a^{\gamma_1} U)^m$ ,  $m < n$ , where  $U$  begins and ends in a  $b$ -symbol. Since  $\sigma_b(R_i) = 0$ ,  $U$  is  $b$ -alternating. But then for  $T_1 \equiv a^{\gamma_1} B_1 a^{\gamma_1}$  to be a subword of  $\phi(S)$  either  $a^{\gamma_1} B_1$  or  $B_1 a^{\gamma_1}$  is a subword of  $S$ , so  $a^{\gamma_1} B_1$  (say) is a piece since  $R_1$  is  $b$ -alternating and so is not a conjugate of  $S^{\pm 1}$ . Finally  $p(T_1) < 2+1 < 5$ . But this is a contradiction and so the proof is complete.

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**Proof of 2.13.** We will show that there is an algorithm to decide for any pair  $\langle a, b; R \rangle$ ,  $\langle a, b; \bar{R} \rangle \in J$  whether or not there is an automorphism  $\phi$  of  $F_2$  such that  $\langle R \rangle^{F_2} = \langle \phi(\bar{R}) \rangle^{F_2}$ . The theorem then follows by Proposition C. Suppose there is a  $\phi \in \text{Aut}(F_2)$  with  $\langle R \rangle^{F_2} = \langle \phi(\bar{R}) \rangle^{F_2}$ . Then by 2.2, up to an inner automorphism,  $\phi : a \mapsto a^\epsilon$ ,  $b \mapsto a^\lambda b^{\epsilon'}$ . Let  $R_0 = R$ ,  $R_1 = \chi_a R$ ,  $R_2 = \chi_b R$ ,  $R_3 = \chi_a \chi_b R$  and define  $\bar{R}_i$  similarly.  $R_i^*$  and  $\bar{R}_i^*$  satisfy the conditions on  $R$  in the theorem. Clearly there is an  $i$  and a  $j$  with  $\langle R_i \rangle^{F_2} = \langle \eta^\lambda \bar{R}_j \rangle^{F_2}$ . We will show that this must hold with  $|\lambda| \leq 2k$ , in which case the solution to the word problem for  $\langle a, b; R_i \rangle$  and  $\langle a, b; \bar{R}_j \rangle$  enables us to tell whether  $\eta^\lambda(\bar{R}_j) \subseteq \langle R_i \rangle^{F_2}$  and  $\eta^{-\lambda}(R_i) \subseteq \langle \bar{R}_j \rangle^{F_2}$ , providing us with the required algorithm.

Suppose that  $\langle R_i \rangle^{F_2} = \langle \eta^\lambda \bar{R}_j \rangle^{F_2}$  for some  $\lambda$  with  $|\lambda| > 2k$ , but not for any  $\lambda$  with  $|\lambda| \leq 2k$ . From now on, denote  $R_i$  by  $R$  and  $\bar{R}_j$  by  $\bar{R}$ . If  $a^\alpha \in \bar{R}^*$  for some  $\alpha > 0$ , then  $\eta^\lambda$  has the same effect on  $\langle R \rangle^{F_2}$  as  $\eta^\mu$  where  $\mu \equiv \lambda \pmod{\alpha}$  and  $0 \leq \mu < \alpha$ . Similarly if  $a^\alpha \in R^*$ . Let  $U$  be a shortest element of  $R^* \Delta \bar{R}^*$ . Without loss of generality we may assume that  $U \in R^*$ , and that  $U \equiv S_1 a^{\alpha_1} S_2 \dots S_n a^{\alpha_n}$  where  $S_i$  begins in  $b^{\epsilon_i}$  and ends in  $b^{\delta_i}$ , the exponents of  $b$  in  $S_i$  alternate between  $\pm 1$ ,  $|\delta_i| = |\epsilon_i| = -\epsilon_1 = 1$  and  $\epsilon_i = \delta_{i-1}$ . Here  $n$  may be 1 in contrast to the proof of 2.10. Now

$$\eta^\lambda(U) = S_1 a_1^{\alpha_1 + \epsilon_1 \lambda} S_2 \dots a_{n-1}^{\alpha_{n-1} + \epsilon_{n-1} \lambda} S_n a_n^{\alpha_n + \bar{\alpha}}, \text{ where } \bar{\alpha} = 0 \text{ if } \delta_n = 1,$$

$\bar{\alpha} = -\lambda$  if  $\delta_n = -1$ . Clearly  $\eta^\lambda(U)$  is cyclically reduced, and so by 1.2

$\eta^\lambda(U) \in \langle \bar{R}^* \rangle^{F_2}$  implies that either  $\eta^\lambda(U) \in \bar{R}^*$  or contains  $m+1$  disjoint  $m$ -remnants,  $T_i$  of  $V_i \in \bar{R}^*$ . By the choice of  $\lambda$ ,  $\eta^\lambda(U) \in \bar{R}^*$  would imply that  $U$  is  $b$ -alternating in which case  $\eta^\lambda(U) = U$ . Since  $U \in R^*$  and  $U \in R^* \Delta \bar{R}^*$ ,  $U \notin \bar{R}^*$ , but then by 1.6 for  $U \in \langle \bar{R}^* \rangle$  it must contain more than  $\frac{1}{2}$  of some element  $V$  of  $\bar{R}^*$  with  $\ell(V) < \ell(U)$ . This clearly cannot happen since  $V$  would belong to both  $R^*$  and  $\bar{R}^*$ , and this would contradict the  $C'(1/8)$  condition on  $R^*$ . Also by the choice of  $\lambda$ ,  $T_i \equiv a^{\gamma_i} B_i a^{\gamma'_i}$  where the  $B_i$  begin and end in  $b$ -symbols, are disjoint subwords of  $U$ , and the exponents on  $b$  in  $B_i$  alternate between  $\pm 1$ .

Suppose one of the  $B_i \rightarrow B_j$ -say, is a piece. Then

$$\begin{aligned} (1-m/8)\ell(V_j) &< \ell(T_j) \\ &\leq \ell(a^{\gamma_j}) + \ell(B_j) + \ell(a^{\gamma'_j}) \\ &< 1 + \ell(V_j)/8 + \ell(V_j)/8 + \ell(V_j)/8 + 1 \\ &= 2 + 3\ell(V_j)/8, \end{aligned}$$

so that  $(1-(m+3)/8)\ell(V_j) < 2$ . In other words

$$\ell(V_j) < 2/(1-(m+3)/8) \leq 8$$

since  $m \leq 3$ . But, since  $B_j$  is a piece, this is impossible. Thus no  $B_i$  is a piece. Now the smallest  $B_i \rightarrow B_l$  say has  $\ell(B_l) < \ell(U)/(m+1)$ , since the  $B_i$  are disjoint subwords of  $U$  written on a circle. So

$$\begin{aligned} (1-m/8)\ell(V_l) &\leq \ell(T_l) \\ &\leq \ell(a^{\gamma_l}) + \ell(B_l) + \ell(a^{\gamma'_l}) \\ &\leq 1 + \ell(V_l)/8 + \ell(U)/(m+1) + \ell(V_l)/8. \end{aligned}$$

If  $\ell(V_L) < \ell(U)$  then  $V_L \in R^* \cap \bar{R}^*$  (by choice of  $U$ ) and since  $B_L$  is a subword of both  $U$  and  $V_L$ ,  $B_L$  would be a piece. So  $\ell(V_L) \geq \ell(U)$ . Consequently

$$(1-m/8)\ell(V_L) < 2 + (2/8+1/(m+1))\ell(V_L)$$

in other words

$$(1 - \frac{(m+2)}{8} - \frac{1}{m+1})\ell(V_L) < 2.$$

This implies that  $\ell(V_L) < 16$ . But then  $|\gamma_L|, |\gamma'_L| \leq 2$  so that

$\ell(V_L)(1-m/8) < 4 + \ell(V_L)/(m+1)$ . Thus  $\ell(V_L)(1-m/8-1/(m+1)) < 4$ . Consequently

$\ell(V_L) \leq 10$ , and since  $\ell(U) \leq \ell(V_L)$ ,  $\ell(U) \leq 10$ . If  $U \equiv Z^n$  for some  $n > 1$  then  $\ell(Z) \leq 5$  and since  $\sigma_b(U) = 0$ ,  $\sigma_b(Z) = 0$ . However if  $Z$  is  $b$ -alternating then so is  $Z^n$  and the only words  $W$  with  $\sigma_b(W) = 0$  and  $\ell(W) \leq 5$  are of the form  $a^\alpha b a^\beta b^{-1}$  up to cyclic permutations, so that if  $U \equiv Z^n$  then  $U$  is  $b$ -alternating. But  $U$  being  $b$ -alternating has already been excluded. It can be shown without too much difficulty that there are no words of length less than 16 which satisfy  $C'(1/8)$ , have zero exponent sum on  $b$  and are not proper powers.

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### §2.3 The Nielsen equivalence classes of some small cancellation groups.

Problem 3 of Pride [1976b] asks when does a small cancellation group have one T-system? Pride suggests that the case  $G = \langle X; R \rangle$ , where  $R$  satisfies  $C'(\lambda)$  for small enough  $\lambda$ , and where all the elements of  $R$  are proper powers, might be worth investigating. In this section we will concentrate on the two-generator case and show:

2.14 THEOREM. Let  $G = \langle a, t; R \rangle$  where  $R^*$  satisfies  $C'(1/24)$ , all elements of  $R$  are proper powers and  $t$ -alternating. Then any generating pair of  $G$  is Nielsen equivalent to a pair of the form  $(a^\mu, t)$  where  $\mu$  is coprime to the order of  $a$ . Moreover  $2GP$  is solvable for  $G$ .

It follows immediately from this that if  $a$  has infinite order and  $R \not\leq F'(a, t)$  then  $G$  has one Nielsen equivalence class, and so in this case  $G$  is Hopfian by Proposition A. That  $G$  is Hopfian is in fact true if  $a$  has finite order. To see this, note that if  $\mu$  is coprime to the order  $k$  of  $a$  then  $\langle b, t; R(b^\beta, t) \rangle$  is a presentation of  $G$  under the mapping  $b \mapsto a^\mu$ ,  $t \mapsto t$ , where  $\beta$  is such that  $\alpha k + \beta \mu = 1$ . Now by 2.5,  $R(b^\beta, t)$  is stable for any  $\beta$ , since all the elements of  $R$  are  $t$ -alternating. If  $\gamma$  is an endomorphism of  $G$ , then since  $G$  has only finitely many T-systems,  $\gamma^n(a, t)$  must be in the same T-system as  $\gamma^m(a, t)$  for some  $m$  and  $n$  with  $m > n$ , so that  $\gamma^{m-n}(\gamma^n(a, t))$  lies in the same T-system as  $\gamma^n(a, t)$ . The Hopficity of  $G$  now follows by an argument similar to that of Proposition A. This argument applies also the groups considered by Pride [1975].

We see that the above argument relied on the fact that  $G$  had finitely many T-systems, each one of which was associated with a presentation  $\langle X; R \rangle$  with  $R$  stable. In this regard I would conjecture that if  $G$  has finitely many T-systems and if one of them is stable (that is, if for one of the T-systems some generating tuple in the T-system has an associated presentation  $\langle X; R \rangle$  with  $R$  stable) then  $G$  is Hopfian.

It follows immediately from 2.14 and Proposition B that if  $R \not\leq F'(a, t)$ , then  $G$  has finitely generated automorphism group. In fact if  $a$  has infinite order then  $\text{Aut}(G)$  is finitely-presented, for in this case  $\text{Aut}(G)$  is generated by the inner automorphism group,  $\eta^*$  and some subgroup of  $\text{sgp}\{\chi_a^*, \chi_b^*\}$ . So that  $\text{Aut}(G)$  is cyclic-by-finite, and so is finitely presented. (For the definition of  $\eta^*$  etc. see Proposition B.)

Let  $J$  be a recursive set of finite group presentations  $\langle a, t; R \rangle$ , where  $R$  satisfies the conditions in 2.14 with  $R \not\leq F'$ . Then  $\text{ISOP}(J)$  is solvable. This follows by 2.13 in the case that  $a$  has infinite



order in  $G$ . In general since  $\langle \eta(R) \rangle^{F(a,t)} = \langle R \rangle^{F(a,t)}$ , the isomorphisms possible are of the form  $a \mapsto a^\beta$ ,  $t \mapsto t^\epsilon$ , where  $\beta$  is coprime to the order of  $a$ . These can be checked using the solution to the word problem.

The result in 2.14 is just a special case of a more general result.

2.15 THEOREM. Let  $S(a,b)$  be a symmetrized subset of  $F_2$  satisfying  $C'(1/12)$ . Suppose all elements of  $S$  are proper powers and  $a^k \in S$  if and only if  $b^k \in S$ . Let  $R(a,t) = S(a, t^{-1}at)$ . Then the conclusions of 2.14 hold for  $G = \langle a, t; R \rangle$ .

Results of a partial nature in the direction of the above theorem have been obtained in Pride [1973, Theorem 2] and Pride [1975, §3.2].

To see how 2.14 follows from 2.15 let  $R$  be as in 2.14. Let  $T \in R$ . If  $T \equiv a^\alpha$  for some  $\alpha$ , let  $\hat{T} \equiv a^\alpha$ ,  $\hat{T}' \equiv b^\alpha$ . If  $T \not\equiv a^\alpha$ , by taking cyclic permutations if necessary, it can be assumed that  $T$  begins in  $t^{-1}$ . Starting from the left, replace subwords of the form  $t^{-1}a^\mu t$  by  $b^\mu$  in  $T$ , to obtain  $\hat{T}$ . Let  $S(a,b)$  be the resulting set of words  $\hat{T}$  and  $\hat{T}'$ . Clearly  $a^\alpha \in S$  if and only if  $b^\alpha \in S$ . Finally we need to see that  $S$  satisfies  $C'(1/12)$ . Since  $\ell(\hat{T})/\ell(T) \geq \frac{1}{2}$  and no piece other than  $b^k$  is formed which was not a piece before, if  $R$  satisfies  $C'(1/24)$  then  $S$  satisfies  $C'(1/12)$ .

In order to prove 2.15 we will make use of the results and techniques developed by Pride [1975]. The following rather technical definitions

are needed (see loc. cit. p.341).

Let  $H$  be a two-generator group, and let  $(c,d)$  be a generating pair of  $H$ . Then  $H$  will be called an  $FE^{\dagger}$  group relative to  $(c,d)$  (or simply an  $FE$  group if  $c$  and  $d$  are understood or if it is immaterial what  $c$  and  $d$  are) if the following conditions are satisfied.

1.  $c$  and  $d$  have equal order.
2. For every integer  $\lambda$ ,  $\text{sgp}\{c^{\lambda}, d^{\lambda}\} \cap \text{sgp}\{c\} = \text{sgp}\{c^{\lambda}\}$  and  $\text{sgp}\{c^{\lambda}, d^{\lambda}\} \cap \text{sgp}\{d\} = \text{sgp}\{d^{\lambda}\}$ .
3.  $\text{sgp}\{c\}$  and  $\text{sgp}\{d\}$  are malnormal in  $H$ .
4. If  $c$  and  $h^{-1}dh$  generate  $H$  then  $h$  is expressible in the form  $d^{\delta}c^{\lambda}$ .

Now suppose  $H = \langle a, b; S \rangle$ , and suppose  $c, d$  are given as words in  $a, b$ . Then  $H$  will be called an  $AFE^{\dagger\dagger}$  group (relative to  $(c,d)$ ) if it is an  $FE$  group and in addition the following hold.

5. The order of  $c$  is known.
6. There is an algorithm to determine for any word in  $a, b$  whether or not it is expressible in the form  $d^{\delta}c^{\lambda}$ .
7. The generalised word problem is solvable for  $\text{sgp}\{c\}$  and  $\text{sgp}\{d\}$  in  $H$ .

We use the obvious notation  $FE(c,d)$  etc. for an  $FE$  group relative to  $(c,d)$ .

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$\dagger$  Free enough

$\dagger\dagger$  The "A" stands for algorithm

The interest in  $FE$  and  $AFE$  groups stems from the following theorem (see Pride [1975, Theorem 3]).

**THEOREM.** *Let  $H = \langle a, b; R \rangle$  and suppose  $H$  is a non-trivial  $FE(c, d)$  group. Let  $G = \langle a, b, t; R, t^{-1}ct = d \rangle$ . Then any generating pair of  $G$  is Nielsen equivalent to a pair of the form  $(t, c^\mu)$  where  $\mu$  is coprime to the order of  $c$ . If  $H$  is an  $AFE(c, d)$  group then the 2GP is solvable for  $G$ .*

Pride has also shown that  $FE$  and  $AFE$  groups may be constructed from others thus extending the class of groups known to be  $FE$  or  $AFE$ .

Let  $H$  be a two-generator group generated by  $(c, d)$ . Suppose

8. *no non-trivial power of  $c$  is conjugate to a power of  $d$ .*

Then  $H$  is called *special*  $(c, d)$ . We use the abbreviations  $SFE$  for *special*  $FE$  etc. Pride's construction is as follows.

*Let  $H = \langle a, b; R \rangle$  and suppose  $a$  has infinite order, and that  $H$  is an  $SFE(a, b)$  group. Let  $G = \langle a, b, t; R, t^{-1}a^\epsilon t = b \rangle$ . Then  $G$  is an  $SFE(a, t)$  group. Moreover if  $H$  is a  $SAFE(a, b)$  group then  $G$  (when presented on  $a, t$ ) is a  $SAFE(a, b)$  group.*

This result gives us an iterative method of producing groups satisfying 2.15 from others. Note that if the order of  $a$  is infinite then  $G$  has one Nielsen equivalence class.

Many examples of  $SAFE$  groups are given in Pride [1975].

It is now clear that Theorem 2.15 will be established once it is shown that  $H$  is an  $AFE(a,b)$  group.

In fact we will show:

**PROPOSITION.** *Let  $H = \langle a, b; R \rangle$  where  $R$  satisfies  $C'(1/12)$ , all elements of  $R$  are proper powers and  $a^k \in R$  if and only if  $b^l \in R$ . Then  $H$  is a SAFE group relative to  $(a,b)$ .*

The proof of this proposition will proceed by a number of lemmas. Some of these lemmas will be proved under weaker assumptions than those of the proposition.

First we show that certain  $C(12)$  groups satisfy Condition 2.

**2.16 LEMMA.** *Let  $G = \langle X, a, b; R \rangle$  with  $R^*$  satisfying  $C(12)$ , and all elements of  $R$  proper powers unless they involve only  $X$  symbols. Suppose also that both  $a$  and  $b$  are pieces. Then  $\text{sgp}\{a^\alpha, b^\beta\} \cap \text{sgp}\{a\} = \text{sgp}\{a^\alpha\}$ .*

**Proof.** It can be assumed, without loss of generality, that  $\alpha, \beta \geq 0$ ,  $\alpha | m_1$  and  $\beta | m_2$  where  $m_1$  and  $m_2$  are the orders of  $a$  and  $b$  respectively. Let  $L$  be a word of the form

$$(*) \quad a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_n} b^{\beta_n}$$

where  $n > 0$ , the  $\alpha_i$  and  $\beta_i$  are integers, non-zero except possibly for  $\alpha_1$  and  $\beta_n$ ,  $|\alpha_i| \leq \frac{m_1}{2}$ ,  $|\beta_i| \leq \frac{m_2}{2}$ , the  $\alpha_i$  divisible by  $\alpha$  and the  $\beta_i$  divisible by  $\beta$ .

Suppose  $a^\gamma = L$  where  $0 \leq \gamma < \alpha$ . We will show that  $L = 1$  and thus  $\gamma = 0$ .

By 1.8  $L$  contains a 3-remnant  $B$  of some  $S \in R^*$ . Let

$B \equiv a^{\alpha_0} b^{\beta_0} a^{\alpha_1} \dots$  where, without loss of generality,  $\alpha_i, \beta_i \neq 0$ . Let

$S' \equiv Z^n$  be a cyclic permutation of  $S$  beginning in  $b^{\beta_0}$ . If  $p(a^{\alpha_0} Z') \geq p(Z^n) - 3$  for some initial segment  $Z'$  of  $Z$ , then

$$\begin{aligned} p(Z^n) - 3 &\leq p(a^{\alpha} Z') \leq p(a^{\alpha_0}) + p(Z') \\ &\leq 2 + p(Z) \leq 2 + \frac{p(Z^n) - 1}{n} + 1 \quad \text{by 1.3 and 1.4} \end{aligned}$$

thus

$$p(Z^n) \left(1 - \frac{1}{n}\right) \leq 6 - \frac{1}{n}$$

and so

$$p(Z^n) \leq \frac{6n-1}{n-1} \leq 11$$

which contradicts the  $C(12)$  condition on  $R^*$ . Therefore  $a^{\alpha_0} Z b^{\epsilon}$  is an initial segment of  $B$ . So all of the  $a$ -symbols and  $b$ -symbols of  $Z$  occur in  $B$ . Thus  $Z$  is a word in  $a^{\alpha}, b^{\beta}$ . Replacing  $Z$  in  $L$  by its complement in  $S'$ , freely reducing and using  $a^{m_1} = 1 = b^{m_2}$  we get a shorter word of the form (\*). Clearly continuing this process we must end up with the empty word, so that  $L \stackrel{G}{=} 1$ , as required.

||

Groups satisfying the lemma satisfy Condition 2 on  $(a, b)$  by symmetry. The  $C(12)$  condition is slightly stronger than is necessary in the above lemma. The  $C(9)$  condition, together with a condition on the minimum power occurring in  $R$  is possible.

We now look at Conditions 3 and 8.

2.17 LEMMA. Let  $G = \langle X, a, b; R \rangle$ ,  $R$  satisfying  $C'(1/6)$ . Let  $c \in \{a, b\}$ . Then an equation  $a^\alpha W c^\beta W^{-1} = 1$  holds only if either  $a^\alpha = c^\beta = 1$  or  $W = a^\gamma c^\delta$  for some  $\gamma, \delta$ .

Proof. Choose  $(|\alpha|, |\beta|, \ell(W))$  minimal in lexicographical ordering, with  $a^\alpha W c^\beta W^{-1} = 1$  contradicting the lemma. Now if  $a^\alpha W c^\beta W^{-1} = 1$  then  $a^\alpha W c^\beta W^{-1}$ , when written on a circle contains certain subwords as described in 1.2. We will show that this is not so. Notice that if  $W = a^\gamma U c^\delta$ , then  $\ell(U) \geq \ell(W)$ . Also  $W$  does not start in an  $a$ -symbol, nor end with a  $c$ -symbol. Clearly  $a^\alpha, c^\beta, W$  do not contain more than  $1/2$  of an element of  $R^*$ .

Firstly, none of  $a^\alpha W, W c^\beta, c^\beta W^{-1}, W^{-1} a^\alpha$  contain more than  $2/3$  of an element of  $R^*$ . For example, suppose  $W \equiv W_1 W_2$  and  $a^\gamma W_1 U \equiv S \in R^*$  with  $\ell(U) < \frac{\ell(S)}{3}$ . Now since  $a^{|\gamma|-1}$  is a piece,  $|\gamma| < \frac{\ell(S)}{6} + 1$ . Thus if  $\ell(S) \geq 3$  then

$$\begin{aligned} \ell(W_1) &> 2\ell(S)/3 - |\gamma| \\ &> (4/6 - 1/6)\ell(S) - 1 \\ &\geq \ell(S)/3. \end{aligned}$$

So  $\ell(W_1) > \ell(S)/3$ . This is true even if  $\ell(S) < 3$ . It now follows that  $\ell(W_1) > \ell(U)$ . But  $W = a^{-\gamma} U^{-1} W_2$  and  $\ell(U^{-1} W_2) < \ell(W)$ , which as we have noted cannot be true.

Next, neither  $a^\alpha W c^\beta$  nor  $c^\beta W^{-1} a^\alpha$  contains more than  $5/6$  of an element of  $R^*$ . For example if  $a^\gamma W c^\delta U \equiv S \in R$  with  $\ell(U) < \ell(S)/6$ , then  $\ell(W) > 5\ell(S)/6 - |\gamma| - |\delta|$ , and if  $\ell(S) \geq 6$   $|\gamma|, |\delta| < \ell(S)/6 + 1$ ; so that

$$\begin{aligned} \ell(W) &\geq 3\ell(S)/6 - 2 \\ &> \ell(S)/6. \end{aligned}$$

So that  $\ell(W) > \ell(S)/6$ . This is true even if  $\ell(S) < 6$ . In either

case  $\ell(U^{-1}) < \ell(W)$  and  $W = a^{-\gamma} U^{-1} c^{-\delta}$ , again a contradiction.

Thirdly, neither  $Wc^{\beta}W^{-1}$  nor  $W^{-1}a^{\alpha}W$  contains more than  $5/6$  of an element of  $R^*$ . For example suppose  $W \equiv W_1W_2 \equiv W_3W_4$

$$W_2c^{\beta}W_4^{-1}U \equiv S \in R^* \text{ and } \ell(U) < \ell(S)/6.$$

Then the shorter of  $W_2, W_4$  ( $W_2$ , say) would be a piece and

$$\begin{aligned} \ell(W_2c^{\beta}W_4^{-1}) &= \ell(W_2) + \ell(c^{\beta}W_4^{-1}) \\ &< \ell(S)/6 + 2/3\ell(S) \text{ by the first case} \\ &< 5/6\ell(S) \end{aligned}$$

Finally  $S \equiv a^{\alpha}Wc^{\beta}W^{-1} \notin R^*$ , for then  $W$  would be a piece and

$$\begin{aligned} \ell(S) &= \ell(a^{\alpha}Wc^{\beta}W^{-1}) = \ell(a^{\alpha}Wc^{\beta}) + \ell(W^{-1}) \\ &< 5/6\ell(S) + \ell(S)/6, \text{ by the third case} \\ &= \ell(S). \end{aligned}$$

It is now not difficult to check that the subwords required by 1.2. do not occur in  $a^{\alpha}Wc^{\beta}W^{-1}$ .

||

## 2.18 COROLLARY.

1.  $\text{sgp}\{a\}$  is malnormal in  $G$ .
2. No non-trivial power of  $a$  is conjugate to a power of  $b$ .

Thus Conditions 3 and 8 are satisfied for these groups on  $(a, b)$ .

2.19 LEMMA. Let  $G = \langle a, b; F \rangle$ , where  $F^*$  satisfies  $C'(1/12)$ , and where all elements of  $F$  are proper powers. Let  $W \in G$  and  $Y = \text{sgp}\{a^{\mu_1}, W^{-1}b^{\mu_2}W\}$ . Finally suppose that both  $a$  and  $b$  are pieces. Then  $Y \cap \text{sgp}\{a\} = \text{sgp}\{a^{\mu_1}\}$ . Moreover if  $Y \cap \text{sgp}\{b\} \neq 1$  then  $W^G = b^\beta a^\alpha$  for some  $\alpha, \beta$ .

Proof. Let  $m_1, m_2$  be the orders of  $a$  and  $b$  respectively: if  $a^{\mu_1 G} = 1$ , the lemma follows from 2.17. If  $W^{-1}b^{\mu_2}W^G = 1$ , the lemma is obvious. We may assume that  $W$  does not contain more than  $1/2$  of an element of  $F^*$ . Let  $\widehat{W}$  be a minimal element (with respect to length) of the set  $\{b^\beta W a^\alpha : \alpha, \beta \text{ integers}\}$ . If the lemma holds for  $\widehat{W}$  it holds for  $W$  so, without loss of generality, we will assume  $\widehat{W} = W$ . If  $W$  is empty the lemma follows from 2.16, so we may assume that  $W$  is non-empty. Then  $W$  starts with an  $a$ -symbol and ends in a  $b$ -symbol. Again without loss of generality we may assume that  $\mu_i$  divides  $m_i$  if  $m_i$  is finite. Let  $L$  be a word of the form

$$(*) \quad a^{\alpha_1} W^{-1} b^{\beta_1} W \dots a^{\alpha_n} W^{-1} b^{\beta_n} W$$

where  $n > 0$ , the  $\alpha_i$  and  $\beta_i$  are integers, non-zero except possibly for  $\alpha_1$  and  $\beta_n$ ,  $|\alpha_i| \leq m_1/2$ ,  $|\beta_i| \leq m_2/2$ , the  $\alpha_i$  divisible by  $\mu_1$  and the  $\beta_i$  divisible by  $\mu_2$ .

Suppose  $L = a^{\gamma G}$  where  $0 \leq \gamma < \mu_1$ . We will show that  $L = 1$  and thus  $\gamma = 0$ . A similar argument shows the corresponding result for  $b^{\gamma G} = L$  with  $0 \leq \gamma < \mu_2$ .

Since  $L = a^{\gamma G}$ , by 1.8  $L$  contains a 3-remnant  $B$  of an element  $S \in F^*$ . To fix notation suppose  $S \equiv Z^n \equiv X^{-1}B$ . Since  $B$  is a 3-remnant of  $S$ ,  $\ell(B) > 9/12 \ell(S)$ . We will first show that  $B$  crosses over a subword of the form  $W^{-1}b^\lambda W$  or  $W a^\mu W^{-1}$  in  $L$ . This will imply that  $W$  is a piece.



Suppose by way of contradiction that  $B$  did not contain  $W\alpha^{\mu}W^{-1}$  nor  $W^{-1}b^{\lambda}W$ . By listing all the possible  $B$ 's (up to inverses and interchanging the roles of  $\alpha$  and  $b$ ), we will show that this is impossible.

- (i) If  $B \equiv \alpha^{\alpha}$  or  $W_{\alpha}$ , where  $W_{\alpha}$  is a subword of  $W$ , then  $\ell(B) \leq \ell(S)/2$  by the minimality of  $W$ , and the conditions on  $L$ .
- (ii) If  $B \equiv W_2 \alpha^{\alpha}$  then  $X = W_2^G \alpha^{\alpha}$  with  $\ell(W_2) \leq \ell(X) < 3/12 \ell(S)$  so that  $\ell(B) < 4/12 \ell(S) + 1$ .
- (iii) (a) If  $B \equiv W_2 \alpha^{\alpha} W_4^{-1}$  then the shorter of  $W_2, W_4^{-1} W_4$  say is a piece. By (ii),  $\ell(W_2 \alpha^{\alpha}) \leq 4/12 \ell(S) + 1$ . So  $\ell(W_2 \alpha^{\alpha} W_4^{-1}) \leq 5/12 \ell(S) + 1$ .  
 (b) If  $B \equiv b^{\beta} W \alpha^{\alpha}$ , then  $X = b^{\beta} W \alpha^{\alpha}$  and since  $|\alpha|, |\beta| < \frac{\ell(S)}{12} + 1$  and  $\ell(W) \leq \ell(X) < 3/12 \ell(S)$  then  $\ell(B) < 5/12 \ell(S) + 2$ .
- (iv) If  $B \equiv W_2 (\alpha^{\alpha} W^{-1} b^{\beta})$ , then since  $W_2$  is a piece and using (iii)  $\ell(B) < \frac{5\ell(S)}{12} + 2 + \frac{\ell(S)}{12} = \frac{6\ell(S)}{12} + 2$ .
- (v) If  $B \equiv W_2 (\alpha^{\alpha} W^{-1} b^{\beta} W_3)$ , then since  $W_2$  is a piece and using (iv)  $\ell(B) < 6/12 \ell(S) + 2 + 1/12 \ell(S) = 7/12 \ell(S) + 2$ .

So in all cases  $\ell(B) < 7/12 \ell(S) + 2$ . But  $(1 - 3/12) \ell(S) < 7/12 \ell(S) + 2$  only when  $\ell(S) < 12$  and so all pieces are empty, and thus  $|\alpha|, |\beta| \leq 1$  and  $B$  is a subword of  $b^{\epsilon_1} W \alpha^{\epsilon_2}$ . Thus  $\ell(B) < 2 + 3\ell(S)/12$ , since the part of  $W$  in  $B$  has length  $\leq \ell(X)$ , and

$$9\ell(S)/12 < \ell(B) < 2 + 3\ell(S)/12,$$

which implies that  $\ell(S) < 4$ . The lemma clearly follows if  $\ell(S) < 4$ , for then we have a one-generator group.

It now follows from the above that  $B$  crosses after at least one copy of  $W$  and at least one copy of  $W^{-1}$  in  $L$ . In particular  $W$  is a piece. Recall that

$$L \equiv a^{\alpha_1} W^{-1} b^{\beta_1} W \dots W a^{\alpha_{n_W-1}} W^{-1} b^{\beta_{n_W}}$$

$$\boxed{B(\text{say})}$$

$$S \equiv Z^n \equiv X^{-1} B \in \mathbb{R}^*$$

Let  $B \equiv B' W^\epsilon T$  where  $T$  does not contain an entire copy of  $W^{\pm 1}$  in  $L$ .

$$\boxed{\begin{array}{|c|c|c|} \hline B' & W & T \\ \hline \end{array}}$$

Clearly  $T$  is of the form  $a^{\alpha} W_2^{-1}$  or  $b W_1$ , where the  $W_1$  must omit at least one symbol from  $W$ , so that  $\ell(W_i) < \frac{\ell(S)}{12} - 1$  since  $W$  is a piece. Also  $|\alpha| < \frac{\ell(S)}{12} + 1$  so that  $\ell(T) < \frac{2\ell(S)}{12}$ . Since

$$\ell(B) > 9 \frac{\ell(S)}{12}, \quad \ell(B'W) > \frac{7}{12} \ell(S). \quad \text{Without loss of generality we will}$$

consider the case  $\epsilon=1$ , since the case  $\epsilon=-1$  can be dealt with by analogy.

Let  $\bar{S} \equiv \bar{Z}^n \equiv T X^{-1} B' W$ . Clearly  $\bar{S} \in \mathbb{R}^*$ . Now since  $\ell(\bar{Z}) + \ell(W) < \frac{\ell(\bar{S})}{n} + \frac{\ell(\bar{S})}{12}$

$$\leq 7 \frac{\ell(S)}{12} < \ell(B'W),$$

and since  $\bar{Z}^n$  ends in  $W$ , both  $\bar{Z}$  and  $W\bar{Z}$  occur as terminal segments of  $B'W$ , though the  $W$  in  $W\bar{Z}$  need not (as yet) 'match' up with a  $W$  in  $L$ .

$$L \equiv a^{\alpha_1} W^{-1} b^{\beta_1} W \dots W a^{\alpha_{n_W-1}} W^{-1} b^{\beta_{n_W}}$$

$$\boxed{\begin{array}{|c|c|c|c|} \hline B & & & \\ \hline B' & W & T & \\ \hline W & Z & T & \\ \hline \end{array}}$$

Let  $\Pi$  denote a product of the form  $\prod_i a_i^{\bar{\alpha}_i} W^{-1} b_i^{\bar{\beta}_i} W$ . Then  $\bar{Z} \equiv U \Pi_1$  and  $W\bar{Z} \equiv V \Pi_2 \Pi_1$  where  $U, V$  are of the form  $W_2 b^{\beta_{W_2}} W_1^{-1} b^{\beta_{W_1}}$ .

We will show that  $U$  is empty, so that  $\bar{Z}$  is of the form (\*) and appears in this form in  $L$ . Now replacing  $\bar{Z}$  in  $L$  by  $Z^{1-\gamma}$  and freely reducing in terms of  $a$  and  $b$ , we obtain a word  $L'$ . Since all of  $B$  is cancelled in the free reduction,  $\ell(L) > \ell(L')$ . Now using the relations  $a^{m_1}, b^{m_2}$ ,  $L'$  is equal to a word, of the form (\*), again shorter than  $L$ . Continuing this process we must obtain a word  $\bar{L}$  which is freely equal to  $a^\gamma$ . Thus  $a^\gamma$  is of the form (\*) and so  $\gamma$  is a multiple of  $\mu$ , but  $0 \leq \gamma < \mu$  so that  $\gamma=0$ ; as required.

To show that  $U$  is empty, suppose otherwise. We have  $WU \equiv V\Pi_2$ . If  $\Pi_2$  is empty then clearly  $\ell(V) > \ell(W)$ , and since  $W$  starts with an  $a$ -symbol  $V \equiv W_1^{-1}b^\beta W$ . But  $WU \equiv V\Pi_2 \equiv W_1^{-1}b^\beta W$ , which implies that  $W_1 \equiv W_1^{-1}$  and this is clearly impossible in a free group. So  $\Pi_2$  is non-empty and  $\Pi_2 \equiv \Pi_3 a^\alpha W^{-1}b^\beta W$  (say). Thus  $WU \equiv V\Pi_3 a^\alpha W^{-1}b^\beta W$ . So  $\ell(U) > \ell(W)$  and  $U \equiv U_1 W$  so that  $WU_1 \equiv V\Pi_3 a^\alpha W^{-1}b^\beta$ .

If  $\ell(U_1) < \ell(b^\beta)$ ,  $W \equiv V\Pi_3 a^\alpha W^{-1}b^\delta$ , but then  $\ell(W) = \ell(V\Pi_3 a^\alpha W^{-1}b^\delta) > \ell(W)$  which is clearly contradictory, so  $\ell(U_1) \geq \ell(b^\beta)$  and  $U_1 \equiv U_2 b^\beta$ . Thus  $WU_2 \equiv V\Pi_3 a^\alpha W^{-1}$ . Finally it is clear that  $U_2 \equiv U_3 W^{-1}$ , for otherwise  $W$  would end as  $W^{-1}$  starts. But we have now shown that  $U \equiv U_1 W \equiv U_2 b^\beta W \equiv U_3 W^{-1}b^\beta W$  which is impossible by definition of  $U$ . Thus  $U$  must have been empty as required.

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Thus Condition 4 is satisfied by these groups with respect to  $(a, b)$ .

Next we will show that two-generator  $C'(1/6)$ -groups satisfy Conditions 5, 6 and 7.

2.20 LEMMA. Let  $G = \langle a, b; R \rangle$  be finitely presented with  $R^*$  satisfying  $C'(1/6)$ . Then

- (i) If  $a$  has order  $k$  in  $G$  then  $a^k \in P^*$  and vice versa.
- (ii) The generalised word problem for  $\text{sgp}\{a\}$  is solvable.
- (iii) There is an algorithm to determine for any word in  $a$  and  $b$  whether or not it is expressible in the form  $b^\delta a^\lambda$ .

Proof.

(i) Is fairly obvious from 1.2.

(ii)&(iii) If  $a^{\epsilon} b^{\epsilon'} \in R^*$  the lemma is obvious.

Let  $\widehat{W}$  be a word in  $F_2$ ; we need to decide whether or not  $\widehat{W} a^\alpha b^\beta = 1$  for some  $\alpha, \beta$ . Without loss of generality we may assume that  $\widehat{W}$  does not contain more than  $1/2$  of an element of  $R^*$ . Let  $W$  be the shortest element of the form  $b^\gamma \widehat{W} a^\delta$  which is cyclically reduced. We will show that  $b^\beta W a^\alpha = 1$  only if  $b^\beta W a^\alpha \in R^*$  with  $\ell(b^\beta W a^\alpha) < 8$ . By 1.2 if  $b^\beta W a^\alpha = 1$  then  $b^\beta W a^\alpha$  has  $m+1$  disjoint  $m$ -remnants in it for  $m=1, 2$  or  $3$  or  $b^\beta W a^\alpha \in R^*$ . Since  $W$  does not contain more than  $1/2$  of an element of  $R^*$  we have only three cases

- (a)  $S = b^\beta W a^\alpha \in R^*$ . Then  $\ell(S) = \ell(b^\beta W a^\alpha)$   
 $< \ell(S)/6 + 1 + \ell(W) + \ell(S)/6 + 1 \leq 5/6 \ell(S) + 2$  so  
 that  $\ell(S) < 12$ . But this means that the length of the  
 largest piece is less than or equal to 1, so  
 $|\beta|, |\alpha| \leq 2$ .

$$\ell(S) = \ell(b^\beta W a^\alpha) \leq 4 + \ell(S)/2.$$

In other words  $\ell(S) \leq 8$  as required.

(b)  $a^\gamma b^\delta$  contains more than  $4/6$  of an element  $S$  of  $R^*$ .  
 Then  $4\ell(S)/6 < \ell(a^\gamma b^\delta) < 2 + 2/6\ell(S)$ , so that  
 $\ell(S) < 6$ . Thus no pieces can occur and  $|\gamma|, |\delta| \leq 1$ .  
 We now have  $4/6\ell(S) < 2$ , so that  $\ell(S) \leq 2$ , which is not possible.

(c)  $W_2 a^\gamma$  contains more than  $5/6$  of an element  $S$  of  $R^*$ .  
 We have

$$5/6\ell(S) < \ell(W_2 a^\gamma) < \ell(S)/2 + \ell(S)/6 + 1,$$

so that  $\ell(S) < 6$  which means that  $|\gamma| \leq 1$ ; again this gives us  $\ell(S) \leq 2$ .

Since these three cases are the only ones possible if  $W$  does not contain more than  $1/2$  of an element of  $R^*$ , the proof is complete.

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Thus  $C'(1/6)$  groups satisfy Conditions 5, 6 and 7 with respect to  $(a, b)$ .

By combining 2.16 - 2.20 the proposition now follows, in the case when both  $a$  and  $b$  are pieces. If this condition does not hold it is not difficult to show that either  $G$  is a one-relator group with torsion or the free product of two cyclic groups. In the first case the proposition follows from the work of Pride [1975]. In the second by the properties of free products.

## CHAPTER THREE

In §3.1 we consider the presentation of certain subgroups of groups. In particular, we obtain some positive results concerning Problem D.

In §3.2 we look at the malnormality of subgroups generated by subsets of the generators of certain small cancellation groups.

### §3.1 Presentations of certain subgroups.

In this section we will consider Problem D, giving examples of classes of sets  $W$  and groups  $G$  for which  $\text{sgp } W$  does have the obvious presentation. But first we will give examples to illustrate what can go wrong.

Firstly, if the elements of  $W$ , when considered as elements of  $F(X)$ , do not freely generate a subgroup, then problems arise immediately. For example, if  $W_1 = x_1^2$ ,  $W_2 = x_1$  then the empty word is equal in  $F(X)$  to  $W_1^2 W_2^{-4}$ . But taking  $G = F(X)$ , it is clear that  $\text{sgp } \{W_1, W_2\}$  does not have presentation  $\langle a, b ; a^2 b^{-4} \rangle$  under the mapping  $a \mapsto W_1, b \mapsto W_2$ .

This shows that 'free relations' between the elements of  $W$  must be taken into account. Quite often the conditions we will impose on  $W$  will ensure that no such relations hold. These conditions take the form of 'independence' conditions.

Problems can still arise even if the elements of  $W$  freely generate a subgroup of  $F(X)$ . For instance, let  $G = \langle a, b, c ; (abcb)^2 \rangle$ . Then  $\text{sgp } \{abc, b, cba\} = \langle f, g, h ; (fg)^2, (gh)^2 \rangle$  under the mapping  $f \mapsto abc, g \mapsto b, h \mapsto cba$  and not  $\langle f, g, h ; (fg)^2 \rangle$  which would be the obvious presentation.

A more subtle example is the following. Let  $G = \langle a, b, c ; (abc)^3 \rangle$ . Then  $sgp \{abc, bcab, ca\} = \langle f, g, h ; f^3, (gh)^3 \rangle$  under the mapping  $f \mapsto abc, g \mapsto bcab, h \mapsto ca$  and not  $\langle f, g, h ; f^2 \rangle$  which would be the obvious presentation.

Even for  $G$  a finite group, not all subgroups have the obvious presentation. Let  $G = \langle a, b ; ab^2aba^2, a^2bab^3 \rangle$ , and consider  $sgp \{a^2, bab^3, ab^2ab\}$ ; then the obvious presentation for this is  $\langle f, g, h ; fg, gh \rangle$  under the mapping  $f \mapsto ab^2ab, g \mapsto a^2, h \mapsto bab^3$ . This, however, is infinite cyclic, whereas  $G$  itself is finite (see Johnson [1976, p.95, Ex.1]).

The examples so far have worked because there is a lot of 'overlap' between the elements of  $W$ . We will put conditions on  $W$  which restrict this 'overlap' in various ways.

From now on let  $W = \{W_i : i \in I\}$  and let  $h$  be a set of symbols  $h_i$  with  $|h| = |W|$ . The method of showing that  $sgp W$  has the obvious presentation in the cases we are going to consider, is to show that  $G$  has  $\langle h ; T(h) \rangle$  as a factor and  $h \mapsto W(X)$  under the natural map. By Proposition D we obtain the required result.

If  $G = \langle X ; R \rangle$  and  $R$  can be written in  $F$  as freely reduced words  $T$  in  $W$ , and where no two elements of  $W$  have  $X$ -symbols in common, then  $sgp W$  does have the obvious presentation. The proof of this is an extension of the method in MKS (Ex.15, p.218) where  $|W| = 1$ . In fact,  $G$  is an iterated free product,

$$G = \langle X \setminus \bigcup_i \delta(W_i) \rangle * \langle h ; T(h) \rangle * \left( *_{i} \langle \delta(W_i) ; W_i^{|W_i|} \rangle \right),$$

$$h_i = W_i$$

where  $|W_i|$  is the order of  $h_i$  in  $\langle h ; T(h) \rangle$ .

The first factor consists of those  $X$ -symbols not occurring in elements of  $W$ . It is easy to see that the amalgamated subgroups are isomorphic.

This condition on  $W$  is rather strong. As we have seen in the previous chapters small cancellation hypotheses restrict the overlap of words. We will show the following result.

**3.1 THEOREM.** *Let  $G = \langle X ; R \rangle$  and let  $W$  be a set of words in  $X$  such that no  $V \in W$  is a proper power and if  $U, V \in W$  with  $U$  conjugate to  $V^{\pm 1}$  then  $U \equiv V$ . Suppose that  $W_*$  satisfies  $C(6)$  and  $R$  can be written as a set of freely reduced words  $T$  in  $W$ . Then  $\text{sgp } W$  has the obvious presentation.*

To prove this theorem we use small cancellation theory with 'mixed metrics'.

The idea of mixed metrics came to us from Paul Schupp (communicated by Stephen Pride). It has been used by M.al Janabi in his Ph D thesis. We have recently learned that B. Hurley has also done some work using mixed metrics. We, of course, do not claim any originality for the idea. However, the lemma of 3.3, and its use for proving 3.1 are original.

It is necessary to introduce a lot of terminology similar to that of §1.4. Familiarity with the diagram methods of Schupp [1971] will be assumed.



Suppose that  $K = F(X) * H$ , where  $F(X)$  is free on  $X$ . Then every non-identity element  $g \in K$  has a unique representation in *extended normal form* as  $g = g_1 \dots g_n$  where each of the *letters*  $g_i$  is either a non-trivial element of  $H$  or an  $X$ -symbol, no two adjacent letters come from  $H$  and none are inverses. The integer  $n$  is the *length* of  $g$ , written  $\ell(g)$ .

If  $g = g_1 \dots g_k c_1 \dots c_t$  and  $h = c_t^{-1} \dots c_1^{-1} d_1 \dots d_s$  in extended normal form, where  $d_1 \neq g_k^{-1}$  we say that the letters  $c_1, \dots, c_t$  are *cancelled* in forming the product  $gh$ . If  $g_k$  or  $d_1$  belong to  $F(X)$  then  $f = gh$  has extended normal form  $g_1 \dots g_k d_1 \dots d_s$ . If both  $d_1$  and  $g_k$  belong to  $H$  let  $a = g_k d_1$ . Then  $f = gh$  has extended normal form  $g_1 \dots g_{k-1} a d_2 \dots d_s$ . We say that  $g_k$  and  $d_1$  have been *consolidated* to give a single letter  $a$  in the extended normal form of  $gh$ .

We say that a word  $f$  has *reduced form*  $gh$  if the extended normal form for  $f$  is obtained by concatenating the extended normal forms for  $g$  and  $h$ . Thus there is neither cancellation nor consolidation between  $g$  and  $h$ . We say that  $f$  has *semi-reduced form*  $gh$  if  $f = gh$  and there is no cancellation between  $g$  and  $h$ . Consolidation is allowed.

An element  $f$  of  $K$  with extended normal form  $f = f_1 \dots f_n$  is said to be *weakly cyclically reduced* if  $\ell(f) \leq 1$  or  $f_1 \neq f_n^{-1}$ . It is *cyclically reduced* if it is weakly cyclically reduced and  $\ell(f) \leq 1$  or one of  $f_1$  or  $f_n$  belongs to  $F$ .

A subset  $R$  of  $K$  is called *symmetrized* if every  $S \in R$  is weakly cyclically reduced and every weakly cyclically reduced conjugate of  $S$  and  $S^{-1}$  is also in  $R$ . If  $R$  is a set of weakly cyclically reduced words denote by  $R^*$  the smallest symmetrized set containing  $R$ .

A word  $B$  is called a *piece*(relative to  $R$ ) if  $R$  contains distinct elements  $S_1$  and  $S_2$  with semi-reduced forms  $S_1 = BT_1$  and  $S_2 = BT_2$ . If no  $S \in R$  is the product of fewer than  $p$  pieces then  $R$  is said to satisfy  $\hat{C}(p)$ .

The following can be shown to be true using the methods of Schupp [1971].

- i. Let  $R$  be a symmetrised subset of  $K$ . For each sequence  $S_1, \dots, S_n$  of conjugates of elements of  $R$  there exists a diagram which satisfies (1) and (2) of Schupp [1971].
- ii. The diagram of a minimal sequence is reduced.
- iii. If  $M$  is a reduced  $R$ -diagram, then the label on an interior edge of  $M$  is a piece.

Because of these the geometry of  $R$ -diagrams for  $R$  satisfying  $\hat{C}(6)$  is the same as for  $R$  satisfying  $C(6)$  over a free group, and we have :

3.2 THEOREM. Let  $G = F(X) * \langle h ; S(h) \rangle$

$$\{R(X, h), W(X) = h\}$$

where  $R^* \cup \{h_i^{-1} W_i(X) : i \in I\}^*$  satisfies  $\hat{C}(6)$ . Then

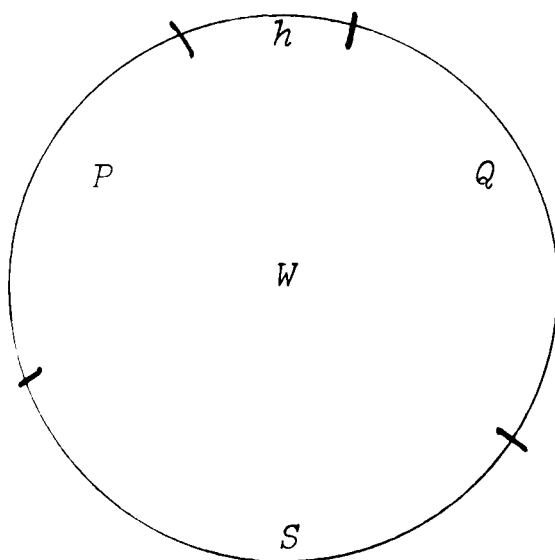
$G = \langle X ; R(X, W(X)), S(W(X)) \rangle$  and  $\text{sgp}_G W(X) = \langle h ; S(h) \rangle$  under the mapping  $h \mapsto W(X)$ .

The proof of 3.1 follows from 3.2 and the next lemma.

3.3 LEMMA. Let  $W = \{W_i : i \in I\}$  be a subset of  $F(X)$  and  $\{h_i : i \in I\}$  be a subset of a group  $H$ . Suppose each  $W_i$  is freely reduced, and that if  $W_i, W_j \in W$  with  $W_i$  conjugate to  $W_j^{\pm 1}$  then  $i = j$ . Suppose also that no  $W_i \in W$  is a proper power. Let  $V = \{W_i h_i : i \in I\}$ . If  $W_*$  satisfies  $C(p)$  then  $V^*$  satisfies  $\hat{C}(p)$ .

Proof. Suppose  $W_i h_i$  when written on a circle can be subdivided into the product of  $q$  pieces. We will show that removing  $h_i$  and closing up the circle gives us a subdivision of  $W_i$  on a circle with at most  $q$  pieces relative to  $W_*$ . We look in turn at the possible pieces making up  $W_i h_i$ . For convenience, write  $W, h$  for  $W_i$  and  $h_i$  respectively.

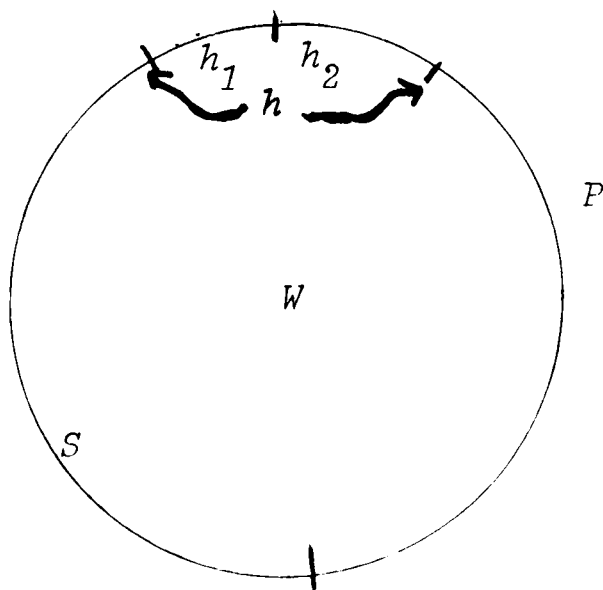
1. A piece involving all of  $h$ .



If  $PhQ$  is a piece relative to  $V^*$  then there are two distinct elements  $PhQS, PhQT$  of  $V^*$  with  $S \neq T$ . So  $PQ$  is a piece relative to  $W_*$ .

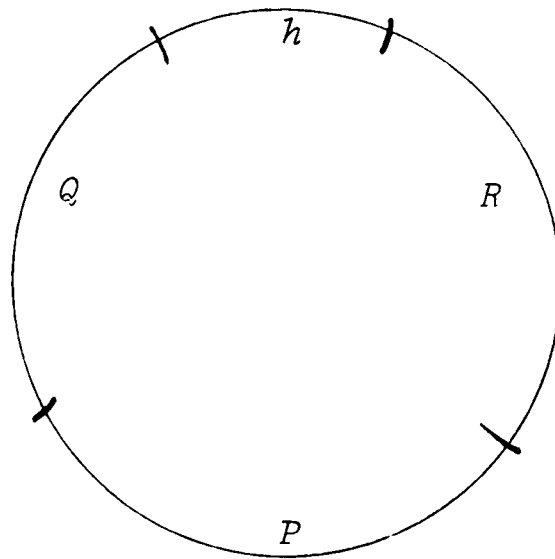
2. A piece beginning with an element of  $H$ .

$$h = h_1 h_2$$



If  $h_2P$  is a piece relative to  $V^*$ , then there are two distinct elements  $h_2PSh_1, h_2PTH_3$  of  $V^*$  with  $Sh_1 \neq Th_3$ . If  $PS \equiv PT$  then  $h_1h_2 = h_3h_2$  by the conditions imposed on  $W$ . So  $PS \not\equiv PT$  and  $P$  is a piece relative to  $W_*$ .

3. A piece not involving an element of  $H$ .



Suppose  $P$  is a piece relative to  $V^*$ . Then there are elements  $PQhR, PSh_j^\epsilon T$  (with  $|\epsilon| = 1, j \in I$ ) of  $V^*$  with  $QhR \neq Sh_j^\epsilon T$ . If  $QR \not\equiv ST$  then  $P$  is a piece relative to  $W_*$ .

Suppose  $QR \equiv ST$ . Then  $R^{-1}(RPQ)R = T^{-1}(TPS)T$ . So since  $RPQ = W_i$  and  $TPS \equiv W_j^\epsilon$ , it follows from the conditions imposed on  $W$  that  $i = j$ .

Thus  $RPQ \equiv (TPS)^\epsilon$ : but  $RPQ$  is conjugate to  $TPS$  and so  $\epsilon = 1$ , for no non-trivial element can be conjugate to its inverse in a free group.

Using the fact that  $QR \equiv ST$  it now follows that  $(S^{-1}Q)RPQ(S^{-1}Q)^{-1} = RPQ$ .

Consequently,  $S^{-1}Q$  and  $RPQ$  are powers of a common element (MKS [Ex.6, p.42]),

which must be  $RPQ$  itself, since  $W$  contains no proper powers. Thus

$S^{-1}Q = (RPQ)^m$  for some  $m$  with  $|m| \geq 1$ . But  $\ell(S^{-1}Q) < \ell((RPQ)^2)$ , so  $|m| = 1$ . But  $RPS = RPQ(S^{-1}Q)^{-1} = (RPQ)^\delta$  where  $\delta = \pm 2$  or  $0$ . However

$RPS \neq 1$  since  $RP, PS$  are freely reduced, and  $\ell(RPS) < \ell((RPQ)^{\hat{c}})$  so  $|\delta| \neq 2$ . This clearly is a contradiction.

||

Another concept which has proved useful when studying the subgroups of groups given in terms of generators and defining relations is  $(a,b)$ -admissibility. Let  $G = \langle a, X, b; R \rangle$ . The tuple  $(U, W, V)$  of freely reduced words is said to be  $(a,b)$ -admissible if  $a$  occurs in  $U$  and each element of  $R$  but not in  $V$  or  $W$ ,  $b$  occurs in  $V$  and each element of  $R$  but not in  $U$  or  $W$ , and finally  $W$  freely generates a subgroup of  $F$  the free group on  $X$ .

Pride [1976a] has proved the following theorem.

**3.4 THEOREM.** *Let  $G = \langle a, X, b; R^n \rangle$  where  $n > 1$  and  $R$  is cyclically reduced. Let  $(U, W, V)$  be  $(a,b)$ -admissible. Then either  $\text{sgp}\{U, W, V\}$  is freely generated by  $\{U, W, V\}$  or  $\text{sgp}\{U, W, V\}$  is a one-relator group with torsion. The second possibility arises if and only if some conjugate of  $R$  can be freely expressed in the free group on  $X, a, b$  as a word  $P(U, W, V)$  in  $U, W, V$ , in which case  $\text{sgp}\{U, W, V\}$  has presentation  $\langle f, g, h; P^n(f, h, g) \rangle$  under the mapping  $f \mapsto U$ ,  $h \mapsto W$ ,  $g \mapsto V$ .*

We are going to make two generalisations of  $(a,b)$ -admissibility. The first enables us to retain the conclusions of the above theorem and the second allows us to solve Problem D for various types of presentation.

Let  $G = \langle a, X, b; R \rangle$ . The pair  $(U, W)$  is  $a$ -admissible (relative to  $b$ ) if all elements of  $R$  are cyclically reduced and involve both  $a$  and  $b$ :

$U$  is a freely reduced word in  $a, X$  which involves  $a^\alpha$  and all elements of  $W$  are freely reduced words in  $b, X$  which freely generate a subgroup of  $G$ .

We will establish the following theorem :

3.5 THEOREM. Let  $G = \langle a, X, b; R \rangle$  where either  $R^*$  satisfies  $C(7)$  with all elements of  $R$  powers or  $R$  is a single relator. Suppose  $(U, W)$  is  $\alpha$ -admissible and  $R$  can be written in  $F$  as freely reduced words  $T$  in  $(U, W)$ . Then  $\text{sgp}_G \{U, W\}$  has the obvious presentation.

We will in fact prove something stronger than this but  $\alpha$ -admissibility will be used to generalise the result of Pride previously noted. The properties of an  $\alpha$ -admissible tuple  $(U, W)$  which are needed for the proof of the theorem are :

(1)  $W$  freely generates a subgroup of  $G$ .

This is part of the definition of  $\alpha$ -admissibility.

(2)  $\{U, X\}$  freely generates a subgroup of  $G$ .

This follows from the Freiheitssatz for one-relator groups and the word problem solution for the small cancellation groups of theorem.

Let  $G = \langle X; R \rangle$ . Then  $W$  is  $n$ -admissible if  $W = (W_1, \dots, W_n)$

and  $\text{sgp}_G \{W_i \cup (\delta(W_i) \cap \bigcup_{j=1}^{i-1} \delta(W_j))\}$  is freely generated by these. We

denote by  $W_0$  the set of elements of  $X$  not occurring in elements of  $W$ .

If  $G$  is a small cancellation or one-relator group, satisfying the conditions in 3.5 then an  $\alpha$ -admissible tuple  $(U, V)$  is 2-admissible with  $W_1 = V$  and  $W_2 = \{U\}$ , by (1) and (2) above.

Not all  $n$ -admissible sets are  $\alpha$ -admissible, for let  $G = \langle a, b, c, d; (abcd)^2 \rangle$  and  $W = \{ab, b^2a^2, cd, d^2c^2\}$ . Clearly  $W$  is not  $x$ -admissible for any  $x \in \{a, b, c, d\}$ , since each of  $a, b, c, d$  occurs in two elements of  $W$ . However, let  $W_1 = \{ab, b^2a^2\}$  and  $W_2 = \{cd, d^2c^2\}$ . By the Newman-Gurevič spelling theorem (1.9)  $\text{sgp } \{ab, b^2a^2\}$  is free on  $ab, b^2a^2$  and  $\text{sgp } \{cd, d^2c^2\}$  is free on  $cd, d^2c^2$ . So  $W$  is 2-admissible but not  $x$ -admissible.

The generalisation of 3.5 we are going to show is :

**3.6 THEOREM.** *Let  $G = \langle X; R \rangle$  and suppose  $W$  is  $n$ -admissible for some  $n$ . Suppose that  $R$  can be written in  $F(X)$  as freely reduced words  $T$  in  $W$ . Then  $\text{sgp } W$  has the obvious presentation.*

**Proof.** To make the proof of the theorem transparent we will deal with the case  $n = 2$ . So suppose  $W$  is 2-admissible in  $G = \langle X; R \rangle$ . This means that  $W = W_1 \cup W_2$  with  $W_1$  and  $W_2 \cup (\delta(W_2) \cap \delta(W_1))$  freely generating subgroups of  $G$ . Divide  $h$  up into  $h_1, h_2$  with

$|h_i| = |W_i|$  and consider the groups

$$G_0 = \langle h ; T(h) \rangle$$

$$\begin{aligned} G_1 &= G_0 * \langle \delta(W_1) \rangle & \text{I} \\ h_1 &= W_1 \end{aligned}$$

$$\begin{aligned} \text{and } G_2 &= G_1 * \langle \delta(W_2) \rangle & \text{II} \\ h_2 &= W_2 \end{aligned}$$

It is easy to see that  $G = F(W_0) * G_2$ . Now there are homomorphisms  $G_0 \rightarrow G_1$  and  $G_1 \rightarrow G$  taking  $h_1 \mapsto W_1$  and  $h_2 \mapsto W_2$ . Since  $\text{sgp}_G W_1$  is free on  $W_1$ ,  $\text{sgp}_{G_0} h_1$  is free on  $h_1$ , so the amalgamated subgroups in I are isomorphic. Similarly, the amalgamated subgroups in II are isomorphic and so the homomorphisms are embeddings, as required.

The general case is similar, for

$$G = \langle h; T(h) \rangle * \left( \bigstar_{i=1}^n \langle \delta(W_i) \rangle \right) * F(W_0)$$

$$h_i = W_i$$

and the amalgamated subgroups are of the form

$$\text{sgp}_G \{W_i \cup (\delta(W_i) \cap \bigcup_{j=1}^{i-1} \delta(W_j))\}, \text{ which by assumption are freely}$$

generated by  $W_i \cup (\delta(W_i) \cap \bigcup_{j=1}^{i-1} \delta(W_j))$  and so  $\langle h; T(h) \rangle$  embeds in  $G$

||

We are interested in generalising the result of Pride (see 3.4) and in this connection the following definition is needed.

Let  $R$  be a set of cyclic words on  $X$  and let  $W$  be a set of words  $W_i$  in  $X$  which freely generate a subgroup of  $F(X)$ . Consider the set of cyclic words in  $W_i$  which when rewritten in terms of  $X$  and reduced give elements of  $R$ . Break each word  $U$  in this set to give a set of words  $T$  in  $W$ . Clearly  $T$  depends upon  $R$  and  $W$  but we will suppress this dependence. Also  $T$  depends upon the way the words were broken but  $T^*$  does not.

**3.7 THEOREM.** *Let  $G = \langle a, X, b; R^n \rangle$ ,  $n > 1$ . Let  $W = \{U, V\}$  where  $(U, V)$  is  $a$ -admissible relative to  $b$ . Then  $\text{sgp } W = \langle h; T(h) \rangle$  under the mapping  $h \mapsto W$  and where  $T$  is defined above. In particular, if  $T$  is non-empty then  $\text{sgp } W$  is a one-relator group with torsion.*

**Proof.** Suppose  $U = a$ . If  $\text{sgp } \{a, V\}$  is not free on  $a, V$  there must be a word  $A$  in  $a, V$  which is equal to one in  $G$ . We will show this is derivable from  $T$ . Let  $B$  be the word obtained from  $A$  by freely reducing



in terms of  $a, X, b$ . No  $a$ -symbols are cancelled in this process. Since  $B \stackrel{G}{=} 1$  it contains a subword  $C \equiv (a^{\varepsilon_D})^{n-1} a^{\varepsilon}$  where  $a^{\varepsilon_D} \in F^*$ , by 1.9. Clearly  $D$  can be written in  $F(a, X, b)$  as a freely reduced word  $P$  in  $a, V$  and  $(a^{\varepsilon_P})^{n-1} a^{\varepsilon}$  occurs in  $A$ . Replacing this subword of  $A$  by  $P^{-1}$  and freely reducing in terms of  $a, V$  we get a word  $A'$  in  $a, V$  with less  $a$ -symbols than  $A$ . Continuing this process we must end up with the empty word so  $A$  is derivable from  $T$  as required. Obviously,  $T = 1$  in  $\text{sgp } W$ . This establishes that  $\text{sgp } W = \langle W; T \rangle$ .

Now suppose  $(U, V)$  is  $a$ -admissible relative to  $b$ . If  $\text{sgp}_G \{U, X, b\}$  is free on  $U, X, b$ , then  $\text{sgp}_G \{U, V\}$  is free on  $U, V$ . Now  $(b, \{X, U\})$  is  $b$ -admissible relative to  $a$  in  $G$ . So using the first part,  $\text{sgp}_G \{U, X, b\} = \langle U, X, b; \hat{T} \rangle$  for some  $\hat{T}$ . Clearly since  $R^n$  involves  $a$  and  $b$ ,  $\hat{T}$  must involve  $U, b$  so  $(U, V)$  is  $U$ -admissible relative to  $b$  in  $\langle U, X, b; \hat{T} \rangle$ . Using the first part again, now gives us the desired result once we notice that if  $T_1, T_2 \in \hat{T}$  then  $T_1 \sim T_2^{\pm 1}$  in  $F(W)$ , since  $W$  freely generates a subgroup of  $F(a, X, b)$ .

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**3.8 THEOREM.** *Let  $G = \langle a, X, b; R \rangle$  where  $R^*$  satisfies C(13) with all elements of  $R$  proper powers  $\geq 3$ . Let  $W = (U, V)$  be  $a$ -admissible relative to  $b$ . Finally suppose that  $U$  begins and ends in  $a^{\varepsilon}$ . Then  $\text{sgp}_G W = \langle h; T(h) \rangle$  under the mapping  $h \mapsto W$ .*

The proof of this parallels that of the one-relator case and so will not be given in full here. We need to prove a 'spelling theorem' similar to the Newman-Gurevič one for one-relator groups with torsion (see 1.9).

3.9 LEMMA. Let  $G = \langle a, X; R \rangle$ , where  $R^*$  satisfies  $C(13)$ . Suppose all elements of  $R^*$  involve  $a$  and are powers  $\geq 3$ . Let  $U \stackrel{G}{=} V$  where  $V$  omits  $a$ . Suppose  $U$  and  $V$  are freely reduced and  $U$  involves  $a$ . Then  $U$  has a subword  $C \equiv a^\varepsilon D a^\varepsilon E$ ,  $|\varepsilon| = 1$  which is an initial segment of some  $S \equiv (a^\varepsilon D)^n \in R^*$ , ends in an  $a$ -symbol and contains more than  $1/2$  of the  $a$ -symbols occurring in  $S$ .

Proof. By 1.2  $UV^{-1}$  when freely reduced contains a subword  $\bar{C}$  of some  $\bar{S}^n \in R^*$  with  $\bar{S}^n \equiv \bar{C}B$  and  $\bar{C}$  a 3-remnant, i.e.  $p(B) \leq 3$ . There are two cases to consider.

Case 1.  $n$  even

We claim that  $\bar{C}$  contains at least  $\frac{n}{2} + 1$  copies of  $\bar{S}$  in it. For if not  $B$  would contain  $\frac{n}{2} - 1$  copies of  $\bar{S}$ . But  $4(\frac{n}{2} - 1) = 2n - 4 \geq n$ . (As  $n > 2$ .) So, by 1.4  $p(\bar{S}^n) \leq 4p(\bar{S}^{\frac{n}{2} - 1}) \leq 4p(B) \leq 12$ , contradicting the  $C(13)$  condition on  $R$ .

Case 2.  $n$  odd.

We claim that  $\bar{C}$  contains at least  $\frac{n+1}{2}$  copies of  $\bar{S}$  in it. For if not,  $B$  would contain  $\frac{n-1}{2}$  copies of  $\bar{S}$ . But  $4(\frac{n-1}{2}) = 2n - 2 \geq n$ . So by 1.4  $p(\bar{S}^n) \leq 4p(\bar{S}^{\frac{n-1}{2}}) \leq 4p(B) \leq 12$  again contradicting the  $C(13)$  condition.

Finally let  $C$  be the maximal subword of  $\bar{C}$  beginning and ending in an  $a$ -symbol. Then  $C$  is the required subword.

||

It may be thought that the small cancellation condition in the above lemma is unnecessary and could be dropped without changing the conclusions. We will show that this is not the case. In fact, consider the following statement: Suppose  $G = \langle X, Y; R \rangle$  where each element of  $R$  is cyclically reduced and is a proper power  $\geq 3$ . Suppose also that some  $S \in R$  involves a  $Y$ -symbol. Then any word in  $X$  which defines the identity element of  $G$  belongs to  $\langle R \setminus S \rangle^{F(X \cup Y)}$ .

For a group satisfying the conditions of the lemma, the statement holds. However the following example will show that the statement is false in general.

Let  $G = \langle a, b, c, d; (abc)^3, (dbc)^3, (a^{-1}c^{-1}b^{-1}a^{-1}dbcd)^3 \rangle$ , we will show that  $(ad^{-1})^3 = 1$  in  $G$ , which contradicts the statement.

Since

$$(abc)^3 = 1, a = (abc)^{-2}(bc)^{-1} = (c^{-1}b^{-1}a^{-1})^2(bc)^{-1}$$

and since

$$(dbc)^3 = 1, d = (dbc)^{-2}(bc)^{-1}$$

so that

$$ad^{-1} = c^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1}(bc)^{-1}(bc)dbcd \sim a^{-1}c^{-1}b^{-1}a^{-1}dbcd$$

and

$$(ad^{-1})^3 \sim (a^{-1}c^{-1}b^{-1}a^{-1}dbcd)^3 = 1,$$

as required.

*Sketch of proof of 3.8.* The case  $U = a$  is almost identical to this case for 3.7 with  $C \equiv a^\epsilon D a^\epsilon E$  in place of  $C \equiv (a^\epsilon D)^{n-1} a^\epsilon$ . Replacing

the corresponding subword by its complement in  $(a^{\varepsilon_D})^n$  gives us a word with less  $a$ -symbols. The general case is also similar to that of 3.7 except that the extra condition on  $U$  ensures that the group obtained in the first part satisfies  $C(13)$ .

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The power condition of the theorem cannot be weakened to  $n \geq 2$ , for let  $G = \langle a, b, c, d, e ; (aW(b, c))^2, (a^{-1} V(d, e))^2 \rangle$ . Clearly  $G$  can be chosen to satisfy any  $C(p)$  condition. Now  $\text{sgp}_G \{a, WV, VW\}$  is not free for  $aWVa^{-1} VW \sim WaWa a^{-1} Va^{-1} V \stackrel{G}{=} 1$ . But clearly no element of  $R^*$  can be written  $F$  as a word in  $a, WV, VW$ .

The other results of Pride [1976a] have similar generalisations since they again depend upon the Newman-Gurevič<sup>✓</sup> spelling theorem.

### §3.2 The malnormality of certain subgroups of certain small cancellation groups.

Recall that if  $H$  is a subgroup of  $G$ , then  $H$  is malnormal in  $G$  if whenever  $g^{-1}hg \in H$  with  $g \in G$  and  $h \in H$  then  $g \in H$ , or  $h = 1$ .

The properties of HNN extensions (or free products with amalgamation) of groups are greatly simplified if the associated (or amalgamated) subgroups are malnormal in the base (or factors). We have made use of this fact in Chapter Two. Pride has also made use of this in studying subgroups of one-relator groups with torsion. The properties of free products of two groups with a malnormal amalgamated subgroup, have been studied by Karrass and Solitar [1971b]. They give a great

deal of information about the subgroup structure of such groups in terms of subgroups of its factors. In particular, they show that a two-generator subgroup of one of these groups is the free product of two cyclic groups or is isomorphic to a subgroup of a factor.

B. Newman [1973] has shown that any subset of generators in a one-relator group with torsion is malnormal. This has turned out to be an extremely important result in dealing with one-relator groups with torsion. (See for example Newman [1973] and Pride [1975]).

In this section we are concerned with obtaining a similar result to Newman's for small cancellation groups. Before doing this, however, we give some examples to show what might go wrong.

Let  $G_i = \langle x_1, \dots, x_i ; x_1^2 \dots x_i^2 \rangle$ . Then  $\{x_1^2 \dots x_i^2\}^*$  satisfies  $C'(\lambda), \lambda > \frac{1}{2i}$ . But since  $x_1^2 = (x_2^2 \dots x_i^2)^{-1} \in \text{sgp} \{x_2, \dots, x_i\}$  and  $x_1 x_1^2 x_1^{-1} = x_1^2$  with  $x_1 \notin \text{sgp} \{x_2, \dots, x_i\}$ . So  $\text{sgp} \{x_2, \dots, x_i\}$  is not malnormal in  $G$ .

Thus we see that it is not always true that a subset of the generators of a small cancellation group generates a malnormal subgroup. To exclude the above example we will insist that all defining relations are powers. This seems reasonable by analogy with Newman's result. However, even this is not enough, for let

$$G = \langle a, b, c, d, e ; (aW(b, c))^2, (a^{-1}V(d, e))^2 \rangle.$$

Then  $G$  can be chosen so as to satisfy any  $C'(\lambda)$  condition. Now

$a^{-1}VW a W V \stackrel{G}{=} a^{-1}V a^{-1}V \stackrel{G}{=} 1$ . But  $a \notin \text{sgp} \{b, c, d, e\}$ , so  $\text{sgp} \{b, c, d, e\}$  is not malnormal in  $G$ .

In contrast to the last example we will prove the following.

**3.10 THEOREM.** *Let  $G = \langle X, Y ; R \rangle$  with  $X, Y$  disjoint,  $R^*$  satisfying  $C'(1/6)$  and all elements of  $R$  which involve both  $X$ -symbols and  $Y$ -symbols are powers  $\geq 3$ . Then  $\text{sgp } Y$  is malnormal in  $G$ .*

In fact it is possible to show that the theorem is true if at most one of the elements of  $R$  (up to inverses and cyclic permutations) which involves  $X$ -symbols and  $Y$ -symbols is a square. But the proof becomes rather tedious and so we will content ourselves with the proof of the theorem as stated, just noting that the previous example is minimal in some sense.

**Proof.** Let  $A, B$  be freely reduced words in  $Y$ , and  $U$  be a freely reduced word in  $X$ ,  $Y$  not belonging to  $\text{sgp } Y$ . Suppose  $A \neq 1$  and  $B \neq 1$  in  $G$ . We will show that  $UAU^{-1}B \neq 1$  in  $G$ . If  $UAU^{-1}B \stackrel{G}{=} 1$  then let  $U, A, B$  be chosen so that  $(\ell(UAU^{-1}B), \ell(U), \ell(A))$  is minimal. Then  $U$  begins and ends in an  $X$ -symbol. Moreover the following hold.

- (1)  $U, A, B$  do not contain  $> \frac{1}{2}$  of an element of  $R^*$ .
- (2) If  $U \equiv VW$ ,  $A \equiv A_1A_2$  and  $S \equiv WA_1C^{-1} \in R^*$  then  $\ell(C) \geq \ell(S)/3$ .

For otherwise, since  $\ell(A_1) < \ell(S)/3$ , then  $\ell(C) < \ell(W)$  and

$$1 = UAU^{-1}B = VWA_1A_2A_1^{-1}W^{-1}V^{-1}B = (VC)A_2A_1(VC)^{-1}B$$

contradicting the minimality of  $(\ell(UAU^{-1}B), \ell(U), \ell(A))$ .

(3) If  $U \equiv V_1 W_1 \equiv V_2 W_2$  and  $S \equiv C^{-1} W_1 A W_2^{-1} \in R^*$  then  $\ell(C) > \ell(S)/6$ .

For without loss of generality we may assume that  $\ell(W_2) \leq \ell(W_1)$ . So  $W_2$  is a piece and  $\ell(W_1 A W_2^{-1}) \leq \ell(W_1 A) + \ell(W_2^{-1}) < 5\ell(S)/6$ , by (2).

(4) If  $A \equiv A_1 A_2$ ,  $B \equiv B_1 B_2$  and  $S \equiv C A_2 U^{-1} B_1 \in R^*$  then  $\ell(C) \geq \ell(S)/6$ .

For otherwise, since  $\ell(A_2), \ell(B_1) < \ell(S)/3$ ,  $\ell(U) > \ell(S)/6 > \ell(C)$ . But

$$1 = UAU^{-1} B \sim B_1^{-1} U A_2^{-1} A_2 A_1 A_2 U^{-1} B_1 B_2 B_1 = C A_2 A_1 C^{-1} B_2 B_1$$

contradicting the minimality of  $(\ell(UAU^{-1}B), \ell(U), \ell(A))$ .

(5)  $S \equiv UAU^{-1} B \notin R^*$ .

As  $\ell(U) < \ell(S)/6$  since  $U$  would be a piece. So

$$\ell(UAU^{-1}B) = \ell(U) + \ell(AU^{-1}B) < (5/6 + 1/6)\ell(S) = \ell(S), \text{ by (4).}$$

Conclusions similar to the above with the roles of  $A$  and  $B$  interchanged follow similarly.

By 1.2  $UAU^{-1}B$  when written on a circle contains certain disjoint subwords of elements of  $R^*$ . It is not difficult to see that this cannot be the case by (1) - (5).

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